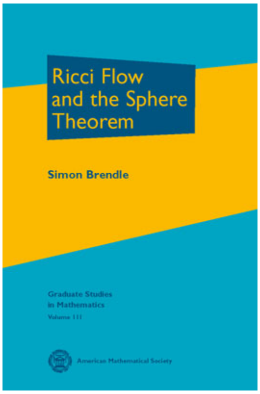
Jahresber Dtsch Math-Ver (2011) 113: 49–54
DOI 10.1365/s13291-011-0014-y

BUCHBESPRECHUNG

Simon Brendle: “Ricci Flow and the Sphere Theorem”

Am. Math. Soc. 2010, 176 pp.

Klaus Ecker



This monograph gives an account of the recent proof of the famous *Differentiable Sphere Theorem* due to the Simon Brendle and Richard Schoen. Their method relies on Hamilton's Ricci flow for metrics on manifolds. Let us first review the relevant background material following the first two chapters of this book.

For a smooth Riemannian manifold (M, g) we define the *Riemann curvature tensor* by

$$-R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$$

for vector fields X, Y, Z, W on M where ∇ denotes the unique way of covariantly differentiating vector fields in the direction of other vector fields (this rule produces again vector fields and is invariant under coordinate transformations) which is compatible with the metric (a kind of product rule condition) and torsion-free, that is $[X, Y] = \nabla_X Y - \nabla_Y X$. Hence the curvature tensor in some sense measures the defect in commuting second derivatives of vector fields. The third term on the right hand side is included to ensure that R is a tensor field in all four entries. The curvature tensor satisfies the symmetry condition

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Z, W, X, Y).$$

Therefore, one may view it as a symmetric bilinear form on the space of 2-forms on M. The *curvature operator* $\mathcal{R}$ is defined on pairs of 2-forms by

$$\mathcal{R}(X \wedge Y, Z \wedge W) = R(X, Y, Z, W).$$

K. Ecker (✉)
FU Berlin, Germany
e-mail: kecker@zedat.fu-berlin.de

(M, g) is said to have *nonnegative (positive) curvature operator* if $\mathcal{R}(\omega, \omega) \geq (>)\ 0$ for all 2-forms ω on M.

Probably the most important curvature quantities are the *sectional curvatures* of M. These can be intuitively understood as the Gauß curvatures of two-dimensional cross-sections of M where the Gauß curvature at a point of M can in turn be determined by measuring the deviation of the angle sum of infinitesimal geodesic triangles from 180 degrees inside that cross-section near that point. The Gauß curvature of a 2-sphere is positive as this angle sum exceeds π, on the plane equal to zero and on a saddle surface negative. In terms of the curvature tensor, the sectional curvature in the 'direction' of a 2-plane Π inside the tangent space of M is defined by

$$K(\Pi) = \frac{R(X, Y, X, Y)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

where X, Y is a basis of Π and $\langle\cdot,\cdot\rangle$ denotes the metric g. One easily checks that the sectional curvature is independent of the choice of basis. The sectional curvatures of the standard unit sphere all equal 1, the ones for hyperbolic space equal -1 and the product $S^2 \times \mathbb{R}$ has sectional curvatures equal to $1, 0$ and 0 everywhere. In three dimensions, the condition of having nonnegative (positive) sectional curvatures is equivalent to the condition of nonnegative (positive) curvature operator. However, in higher dimensions the latter is a much stronger condition. For instance, the sectional curvatures of the 4-manifold $\mathbb{C}P^2$ (complex projective space) all have values $1/4$ and 1 but $\mathbb{C}P^2$ does not have positive curvature operator. The *Ricci tensor* of (M, g) is defined by

$$Ric(X, Y) = \sum_{k=1}^{n} R(X, e_k, Y, e_k)$$

and the *scalar curvature* by

$$scal = \sum_{k=1}^{n} Ric(e_k, e_k)$$

where the $\{e_1, \ldots, e_n\}$ form a local orthonormal frame on M. Being averages of the curvature tensor, these quantities contain less information about the manifold. However, in three dimensions the Riemann tensor can be recovered from the Ricci tensor, so no information is lost there.

In the book by Brendle, several additional curvature conditions on Riemannian manifolds are considered. One is the condition of a *strictly δ-pinched manifold in the global sense* for $\delta \in (0, 1)$. This refers to the condition that the sectional curvatures of (M, g) lie in the interval $(\delta, 1]$. For the weak form of the condition, they are required to lie in the closed interval $[\delta, 1]$ instead. Thus $\mathbb{C}P^2$ is weakly but not strictly $1/4$-pinched in the global sense. Brendle also defines the condition of (M, g) being *strictly δ-pinched in the pointwise sense*. Here, the inequality $0 < \delta K(\Pi_1) < K(\Pi_2)$ for all points $p \in M$ and all 2-planes Π_1, Π_2 inside the tangent space T_pM is required. For the weak form of this definition only $\leq$ is assumed above. These conditions control ratios of sectional curvatures across M rather than their range of values.

The famous *Topological Sphere Theorem* by Berger [1] and Klingenberg [6] states that every compact, simply connected Riemannian manifold which is strictly $1/4$-

pinched in the global sense must be homeomorphic to the standard sphere S^n. In 1956, Milnor [8] had shown that there exist smooth manifolds which are homeomorphic but not diffeomorphic to S^7 (so-called exotic 7-spheres). This shows that additional conditions have to be imposed in order for a manifold to be diffeomorphic to S^n. A natural question is whether the condition of (M, g) being strictly 1/4-pinched in the global sense would be sufficient to guarantee this. This has become a longstanding open problem known as the *Differentiable Sphere Theorem (Conjecture).*

In 1982, Hamilton [4] introduced an evolution method for metrics on Riemannian manifolds, the so-called Ricci flow, and used this to derive two important topological classification results: In three dimensions, the condition of positive Ricci curvature on a compact manifold (M, g) as well as simple connectedness are sufficient to guarantee that M is diffeomorphic to S^3 while in four dimensions the Ricci curvature condition is replaced by the positivity of the curvature operator [5]. In 2008, Böhm and Wilking [2] could show, again using Ricci flow, that positivity of the curvature operator is a sufficient condition in all dimensions for a manifold to admit a metric with all sectional curvatures equal to 1 and hence to be diffeomorphic to S^n up to quotients by certain finite group actions.

Hamilton's Ricci flow for a 'time-dependent' family of metrics $g(t)$ on a manifold M is defined by the evolution law

$$\frac{\partial}{\partial t} g(t)(X, Y) = -2Ric_{g(t)}(X, Y)$$

where we start with some metric $g(0) = g_0$. This evolution equation implies an equation for the curvature tensor which simplifies significantly if we consider instead the curvature operator and write everything in a suitable time-dependent frame. The curvature operator then satisfies the equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{R} = \mathcal{R}^2 + \mathcal{R}^{\#} \equiv \mathcal{Q}(\mathcal{R})$$

which is a reaction-diffusion equation. Here Δ is the Laplace-Beltrami operator applied to tensors on M. The right hand side $\mathcal{Q}(\mathcal{R})$ is quadratic in $\mathcal{R}$ and has a certain algebraic structure encapsulated in the expression $\mathcal{R}^{\#}$. Understanding this is crucial for a detailed analysis of the behaviour of this equation. The reaction term can cause singularities in finite time for certain initial metrics. For example, a round S^3 and some 3-manifolds close to it 'contract' in finite time T. To understand what is happening asymptotically to the geometry one considers instead a volume normalized Ricci flow (or any other time-dependently rescaled flow which ensures bounded curvatures) for closed manifolds. For certain initial metrics, the diffusive effect of the Ricci flow dominates after normalization. In fact, for simply connected closed 3-manifolds (M, g_0) of positive Ricci curvature Hamilton [4] showed in 1982 that this curvature condition is preserved and that the volume normalized Ricci flow converges smoothly to the standard S^3. Therefore, the initial manifold must have been diffeomorphic to the 3-sphere.

One of the main tools used in the analysis is Hamilton's maximum principle for the curvature tensor [5] which roughly states the following: Let F be a closed, convex and $O(n)$-invariant subset of the set of all multilinear 4-tensors on $\mathbb{R}^n$ which have the symmetries of the curvature tensor (actually we freely alternate between the usage of the curvature tensor and the curvature operator here) and in addition satisfy a permutation relation known as the first Bianchi identity (which the curvature tensor of the manifold also satisfies). Suppose that F is invariant under the ordinary differential equation

$$\frac{d}{dt}\mathcal{R} = \mathcal{Q}(\mathcal{R})$$

where $\mathcal{Q}(\mathcal{R})$ appears in the evolution equation for $\mathcal{R}$ valid under Ricci flow. Moreover, suppose that M is a compact manifold and $g(t)$, $t \in [0, T)$ solves the Ricci flow. Assume furthermore that the curvature operator of $(M, g(0))$ satisfies $\mathcal{R}(p, 0) \in F_{(p,0)}$ for all $p \in M$. Then $\mathcal{R}(p, t) \in F_{(p,t)}$ for all $t \in [0, T)$.

Another crucial result of Hamilton requires an additional condition for the sets F considered in the statement of the maximum principle. This involves the notion of a *pinching set* which is somehow related to our conditions of pointwise curvature pinching considered earlier, in that it involves a condition controlling the ratio of sectional curvatures: In fact, in addition to the conditions on F required for the maximum principle we also ask that for each $\delta \in (0, 1)$ the set of $\mathcal{R} \in F$ which are not weakly δ-pinched (we do not use the term 'in the pointwise sense' here as F is a fixed set in the space of multilinear 4-tensors on $\mathbb{R}^n$) is a bounded set. This roughly amounts to saying that the inequality in the pinching definition holds for every $\delta \in (0, 1)$ up to some constant depending on this δ. Such sets F are called *pinching sets*.

Hamilton then showed in [4] that if M is a compact manifold of dimension $n \geq 3$ with a metric g_0 of positive scalar curvature, if there is a pinching set F such that the curvature operator of g_0 is contained in F at all points in M and if $g(t)$ is a solution of Ricci flow on a maximal time interval $[0, T)$ with initial metric g_0, then as $t \to T$, the metrics $\frac{1}{2(n-1)(T-t)}g(t)$ converge in C^∞ to a metric having all sectional curvatures equal to 1. The latter can be topologically classified. In fact, they are equal to S^n up to quotients by finite group actions. This convergence result strongly relies on derivative estimates for the curvature tensor which are derived in Chapter 3 of this book.

All convergence results on Ricci flow starting with (M, g_0) where g_0 satisfies certain curvature conditions such as positivity of the Ricci curvature or the curvature operator were shown by first proving that the curvature condition is preserved by the flow and then finding a suitable pinching set. That the positivity of the curvature operator is preserved by Ricci flow is quite easy to see, since the right hand side expression of the ordinary differential equation considered for the maximum principle satisfies $\mathcal{Q}(\mathcal{R}) \geq 0$ whenever the curvature operator vanishes, so the time-derivative of $\mathcal{R}$ cannot be negative definite then. This in turn implies that $\mathcal{R}$ can never become negative definite if it is initially positive definite.

However, finding a suitable pinching set is usually very involved and relies on a subtle analysis of the algebraic structure of $\mathcal{Q}(\mathcal{R})$. In [5], Hamilton discovered a certain Lie algebra structure and used this to study the case $n = 4$. Böhm and Wilking [2] found additional structures which eventually let them settle the case of positive curvature operator in all dimensions. The book by Brendle gives a very detailed account of all these developments.

The main theorem of the book (proved in Chapters 7 and 8) is the Differentiable Sphere Theorem due to Brendle and Schoen. To explain this we have to introduce yet another curvature condition which plays an important role in their work and also in the work of Nguyen [9]. This condition also features prominently in the paper by Micallef and Moore [7] (it was actually introduced there) which contains one of the first topological classification theorems for 4-manifolds. We say that a manifold (M, g) has *nonnegative isotropic curvature* if the condition

$$\begin{aligned}&R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3)\\&\quad + R(e_2, e_4, e_2, e_4) - 2R(e_1, e_2, e_3, e_4) \geq 0\end{aligned}$$

holds for all orthonormal 4-frames $\{e, e_2.e_3.e_4\}$ on M. There is a more natural way of stating this condition which uses a complexification of the tangent space. In Chapter 7 of Brendle's book, relations between this and other curvature conditions are derived which are stated in a useful flow chart on page 100. It is important for the solution of the Differentiable Sphere Theorem that the condition of $1/4$-pinching in the pointwise sense implies both the conditions of nonnegative scalar curvature (actually even nonnegative sectional curvature) which is required for the initial metric in the pinching set theorem of Hamilton stated above and the condition of nonnegative isotropic curvature.

The main breaktrough due to Brendle and Schoen [3] and independently Nguyen [9] has been to show that the condition of nonnegative isotropic curvature is preserved by Ricci flow. This is a remarkable achievement which uses very subtle algebraic arguments combined with identities obtained by calculating the first and second variation of the isotropic curvature expression with respect to variations of the coordinate frames at a minimum point of this expression. In particular, important information for $\mathcal{Q}(\mathcal{R})$ is obtained at points where the isotropic curvature condition 'reaches' zero. All this is carried out in Chapter 7. Chapter 8 then establishes the existence of a pinching set as required by Hamilton in order to guarantee convergence of the rescaled flow (as defined above) to a metric with sectional curvatures equal to 1 everywhere. Important here is the observation that the set of curvature operators with nonnegative curvature form a closed, convex cone inside the set of all curvature operators but additional conditions relating the Ricci curvature to the scalar curvature have to be imposed to arrive at the final version of the pinching set.

The later chapters of this monograph contain further new topological classification results for manifolds satisfying a variety of curvature conditions. The book concludes with an interesting problem section. This is an excellent self-contained account of exciting new developments in mathematics suitable for both researchers and students interested in differential geometry and topology and in some of the analytic techniques used in Ricci flow. I very strongly recommend it.

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