COMMENTS, ERRATA AND EXTENSIONS FOR THE MONOGRAPH 'REGULARITY THEORY FOR MEAN CURVATURE FLOW'

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Several general comments: *1. All statements in this monograph actually hold for properly immersed rather than only properly embedded solutions, except for the results of Brian White on mean convex mean curvature flow which are described in chapter 5.*

2. In order to make the spelling convention consistent throughout the monograph change the word 'generalize' to 'generalise' everywhere.

3. Replace 'point-wise' by 'pointwise' everywhere.

p.18, last line and page 19, first line : replace these by the following text:

$$\frac{\partial}{\partial t}|F(p,t) - F'(q,t)|^2 \geq 0$$

for all $t > 0$ and $p, q \in M^n$ such that $|F(p,t) - F'(q,t)| = \mathrm{dist}(M_t, M_t')$ where $F(\cdot, t)$ and $F'(\cdot, t)$ denote the immersions for M_t and M_t' respectively.

p.26: Expand lines 8 - 14 in the following way:

Since

$$Df(x,t) = 2(x - x_1)$$

we compute using (3.2) that

$$\Delta_{M_t} f(x,t) = 2n + 2\vec{H}(x) \cdot (x - x_1).$$

In view of (3.1) we also have

$$\frac{df}{dt}(x,t) = 2n + 2\vec{H}(x) \cdot (x - x_1).$$

Combining these we obtain

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)\big(|x - x_1|^2 + 2n(t - t_1)\big) = 0. \tag{3.4}$$

p. 28, line -9:

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) f = -2(n - \beta)(1 - |\nabla x_{n+1}|^2) \leq 0.$$

p. 33, line -3: The last term in the chain of identities should be

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$$-\frac{|\nabla\phi|^2}{\phi}v^2.$$

p. 34, line 3: The last term should be

$$\frac{3}{2}\frac{|\nabla\phi|^2}{\phi}v^2.$$

p.34, line 10: 'we can apply the proof of the maximum principle as on page 122' instead of 'we can apply the maximum principle'. The necessity for this modification stems from the fact that the differential inequality for $v^2\phi$ only holds inside a subset of M_t, so we cannot directly quote Proposition 3.1.

p.36, line 2: Insert before 'Then': Suppose that the vector field a is well-defined in the neighbourhood of all maximum points of $f\phi$.

p.37, line 2: The first two inequalities should state:

$$|a \cdot \nabla\phi| \leq a_0\sqrt{1+d}\,|D\phi| \leq (1+d)\phi + \frac{a_0^2}{4}\frac{|D\phi|^2}{\phi}.$$

p.44, line 8: Delete the term $a \cdot \nabla f$.

p.46 delete lines 12 - 15, i.e. the paragraph before the statement of Theorem 3.24. Replace it by the following text:

Proposition 3.22 in conjunction with the Arzela-Ascoli theorem applied locally to the immersion maps F_{t_k} for an arbitrary sequence of times $t_k \nearrow t_0$ leads to the following local smooth extension of our solution up to time t_0 near a point x_0 once we have curvature estimates near this point prior to t_0. The exact details of this standard convergence argument are left to the reader.

p.46, line 20: replace ' *time* $t_0 + \epsilon$ *inside* $B_{\rho/4}(x_0)$' by '*time* t_0 *inside* $B_\epsilon(x_0)$'

p.46: omit lines 21 - 25, that is the proof of Theorem 3.24

p.52, line 4: Write M_s instead of M_t in the displayed formula as the t variable has already been used up as a terminal for the time integral.

p.52, line 9: It should say $t \in [t_0 - \rho^2/8n, t_0]$ instead of $2n(t - t_0) \leq \rho^2/4$.

p.56, line - 10: Delete 'a time-dependent'. The reference to the type of functions used in Proposition 4.6 is sufficient here. We will also consider time-independent functions below.

p.57, line 2: Leave out the term $R|D\chi_R|$ as its boundedness is not need for the derivation of line 4.

p.57, line 3: Before' one easily checks' insert 'and since $\frac{\partial\phi}{\partial t} \equiv 0$'.

p.63, line -7: It should be $\sup_{t\in(t_0-\rho^2/2n,t_0)}$ at the beginning of the inequality.

p.74, line -8: It should say '$\beta \in (0, \frac{1}{1+2n})$'.

p.75, line 7: It should say '$\beta \in (0, \frac{1}{1+2n})$'.

p.75, line -7: It should say: $I = (t_0 - \rho_0^2, t_0)$.

p.75, line 12: Write 'in $B_{\rho_0}(x_0) \times (t_0 - \rho_0^2, t_0)$' instead of 'in $B_{\rho_0}(x_0)$ on the time interval $(t_0 - \rho_0^2, t_0)$'.

p.76, line 15: It should say '*Let $f \geq 0$ be...*'

p.76, line 21: Insert after '*holds*': *If f satisfies the equation*

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) f = 0,$$

then the condition $f \geq 0$ is not required.

p.76: Insert after the statement of Proposition 4.25:

Remark. (1) Under the conditions of Proposition 4.25 (taking the above alterations such as $f \geq 0$ into account) we even have the inequality

$$f(x_0, t_0) \leq \frac{c(n)}{\rho^{n+2}} \int_{t_0-\rho^2}^{t_0} \int_{M_t \cap B_\rho(x_0)} f.$$

For technical reasons our proof of Proposition does not yield this L^1 - version of the mean value inequality directly. An additional calculus trick due to Rick Schoen contained in a set of handwritten Berkeley lecture notes which were communicated to us by Robert Bartnik implies this improved version. An exposition can be found in [E2]. We repeat the argument here for the convenience of the reader:

In fact, this argument is copied almost verbatim from [E2]: We start with inequality (i) of Proposition 4.25 in [E1] applied at all points in $C_{\frac{\rho}{2}}(x_0, t_0)$. This implies the inequality

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} f \leq c(n) \left(\frac{1}{\rho^{n+2}} \iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} f^2\right)^{\frac{1}{2}}. \tag{1}$$

In order to streamline our notation, we will denote the parabolic cylinder $B_\sigma(x_0) \times (t_0 - \sigma^2, t_0)$ by $C_\sigma(x_0, t_0)$ and use the abbreviations

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} f \equiv \sup_{(t_0-\sigma^2,t_0)} \sup_{M_t\cap B_\sigma(x_0)} f$$

and

$$\iint_{\mathcal{M}\cap C_\sigma(x_0,t_0)} f \equiv \int_{(t_0-\sigma^2,t_0)} \int_{M_t\cap B_\sigma(x_0)} f.$$

By an adjustment of the cut-off function used in the derivation of (1) (namely by choosing it to be equal to 1 in $C_\sigma(x_0,t_0)$ and equal to 0 outside $C_{\sigma+r}(x_0,t_0)$) we obtain instead

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} f \le c(n)\, r^{-\frac{n+2}{2}} \left(\iint_{\mathcal{M}\cap C_{\sigma+r}(x_0,t_0)} f^2 \right)^{\frac{1}{2}} \tag{2}$$

for all $\sigma, r > 0$ with $\sigma + r \le \rho$. This leads to

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} f \le c(n)\, r^{-\frac{n+2}{2}} \left(\sup_{\mathcal{M}\cap C_{\sigma+r}(x_0,t_0)} f \right)^{\frac{1}{2}} \left(\iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} f \right)^{\frac{1}{2}}. \tag{3}$$

We set

$$v = \frac{f}{\iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} f},$$

where we may assume without loss of generality that the denominator does not vanish. Inequality (3) implies

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} v \le c(n)\, r^{-\frac{n+2}{2}} \left(\sup_{\mathcal{M}\cap C_{\sigma+r}(x_0,t_0)} v \right)^{\frac{1}{2}}. \tag{4}$$

We now define $\sigma_0 = \rho/2$ and $\sigma_{i+1} = \sigma_i + r_i$, where $r_i = \rho \cdot 2^{-i-2}$ for $i \in \mathbb{N} \cap \{0\}$. Inequality (4) then yields for all such i

$$\sup_{\mathcal{M}\cap C_{\sigma_i}(x_0,t_0)} v \le c(n)\, \rho^{-\frac{n+2}{2}} \left(2^{\frac{n+2}{2}} \right)^{i+2} \left(\sup_{\mathcal{M}\cap C_{\sigma_{i+1}}(x_0,t_0)} v \right)^{\frac{1}{2}}. \tag{5}$$

Iterating (5) for i between 0 and $j-1$ with $j \in \mathbb{N}$ and using that $\sigma_i < \rho$ for all i leads to

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} v \le \left(c(n)\, \rho^{-\frac{n+2}{2}} \right)^{\sum_{i=0}^{j} 2^{-i}} \prod_{i=0}^{j} 2^{\frac{(n+2)(i+2)}{2^{i+1}}} \left(\sup_{\mathcal{M}\cap C_\rho(x_0,t_0)} v \right)^{\frac{1}{2^j}}.$$

Letting $j \to \infty$ and allowing for a larger constant $c(n)$, we obtain

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} v \le c(n)\, \rho^{-(n+2)}$$

in view of the identity $\sum_{i=0}^{\infty} 2^{-i} = 2$. By the definition of v, we thus arrive at

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} f \le c(n)\, \rho^{-(n+2)} \iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} f.$$

(2) Note that our proof of Proposition 4.25 implies the conclusion of Proposition 4.25 also for (not necessarily non-negative) Lipschitz continuous functions f with smooth f^2 satisfying an inequality of the form

$$\left(\frac{d}{dt} - \Delta_{M_t} \right) f^2 \le -2\, |\nabla^{M_t} f|^2$$

where the factor 2 on the right hand side is not important as long as we have a negative multiple of $|\nabla^{M_t} f|^2$. An example of such a function is $f = |A|e^{-c_0 t}$ for a solution of mean curvature flow which satisfies $|A| \leq c_0$. Indeed, the evolution equation for $|A|^2$ in Proposition 3.19 (i) in combination with Kato's inequality $|\nabla |A||^2 \leq |\nabla A|^2$ implies

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) |A|^2 \leq 2c_0|A|^2 - 2|\nabla |A||^2$$

and hence

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) \left(|A|^2 e^{-2c_0 t}\right) \leq |\nabla^{M_t}(|A|e^{-c_0 t})|^2.$$

Therefore, under the additional assumption of global boundedness of $|A|^2$ we obtain a local bound of $|A|^2$ in terms of a local space-time integral of $|A|^2$ and a constant depending on time. This has important applications, see for instance [E2]. Of course, this local mean value estimate can also be obtained by applying the above L^1 - version of the mean value inequality directly.

p.77, line -9: insert after 'Lemma 3.14': and $f \geq 0$ or alternatively the identity

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) f = 0,$$

.

p.77, line -6: insert after '$\phi \in C_0^2(\mathbb{R}^{n+1} \times \mathbb{R})$' : 'where the compact support is only assumed in the spatial variables'

p.77, line -2: replace c_ϕ by $c(\phi)$. The symbol c_ϕ was previously used for localization functions.

p.78, inequality (4.24): Replace c_ϕ by $c(\phi)$.

p.81, line 12: replace 'beyond time t_0' by 'up to time t_0'

p.90, statement of Theorem 5.6: The theorem should be more clearly stated as follows (this is actually also what is proved later and corresponds to the original statement proved by B. White (see reference [W3] in [E1]):

There exist constants $\epsilon_0 > 0$ and $c_0 > 0$ such that for any properly immersed solution $\mathcal{M} = (M_t)$ of mean curvature flow in $U \times (t_1, t_0)$ the assumption

$$\Theta(\mathcal{M}, x_0, t_0) \leq 1 + \epsilon_0$$

implies that for some $\rho > 0$ and for all $x \in M_t \cap B_\rho(x_0)$ and $t \in (t_0 - \rho^2, t_0)$ the curvature bound

$$|A(x)|^2 \leq \frac{c_0}{\rho^2}$$

holds. In particular, x_0 is a regular point by Theorem 3.24.

Note that in the statement of this theorem the condition that the solution reaches x_0 at time t_0 is not needed!

p.91, line -2: Should be $B_{\rho_0}(x_0) \times (t_0 - \rho_0^2, t_0)$.

p.91: Omit Theorem 5.7. Its proof contains an error on page 97 in line 9 which was pointed out to us by André Neves. Theorem 5.7 will, however, not be needed for the proof of Theorem 5.4. Its proof will be based on Theorem 5.6 only. This affects the numbering of all statements (theorems, lemmas etc) after Theorem 5.6.

p.92, line 6: Add at the end of the sentence: *where also* $(\tau-\rho^2,\tau)\subset(t_0-\rho_0^2,t_0)$ *for those* τ.

p.92: Delete the last two lines of this page and the first two lines of page 93 and replace them by the statement: We set $(x_0,t_0)=(0,0)$.

p.93, line 5: Delete 'reaches $0\in\mathbb{R}^{n+1}$ at time 0 and'.

p.93, line 9: It should say $\sup_{t\in(-(1-\sigma)^2,0)}$ in the inequality.

p.93, line 12: Delete 'which reaches $0\in\mathbb{R}^{n+1}$ at time 0 and'

p.93, line -8: It should say $\sup_{t\in(-(1-\sigma)^2,0)}$ inside statement (5.6).

p.93, line -7: delete parentheses and text inside them after '$j\to\infty$'.

p.93, lines -7 and -6: Between '$j\to\infty$ and 'In particular, one can find a $\sigma_j\in(0,1]$...' insert:

We may assume without loss of generality that $\gamma_j^2<\infty$ for every $j\in\mathbb{N}$ for this sequence of solutions. Otherwise we construct a new sequence as follows: We only need to alter our solution for the integers j for which $\gamma_j^2=\infty$. For these j and arbitrary $\delta\in(0,1)$ we consider

$$\gamma_{j,\delta}^2=\sup_{\sigma\in[0,1-\delta]}\left(\sigma^2\sup_{(-1,-\delta^2)}\sup_{M_t^j\cap B_{\sqrt{1-\delta^2}}(0)}|A|^2\right).$$

Since our solutions are smooth prior to time 0 we have $\gamma_{j,\delta}^2<\infty$ for each $j\in\mathbb{N}$ and $\delta>0$. Since $\gamma_j^2=\infty$ we can find a $\delta_j\in(0,1)$ such that $\gamma_{j,\delta_j}^2\geq j$. For fixed j the quantity $\gamma_{j,\delta}^2<\infty$ is nonincreasing as a function of δ and therefore we may also assume that $\delta_j\searrow 0$ as $j\to\infty$. We now define a new solution by

$$\hat{M}_s^j=\frac{1}{\sqrt{1-\delta_j^2}}M^j_{(1-\delta_j^2)s-\delta_j^2}.$$

This solution is again defined in $B_1(0)\times(-1,0)$ but is smooth up to time 0. In particular, the expression $\hat{\gamma}_j^2$ corresponding to this solution is finite.

Inequality (5.5) still holds for the new solution in view of Proposition 4.17 and the rescaling and translation properties of the integral appearing in (5.5). We have to take a slightly larger radius $\rho_j>1$ due to the fact that we have rescaled the original solution but only the fact that $\rho_j>1$ is relevant for our proof.

p.93, line -5: It should say $\sup_{t\in(-(1-\sigma)^2,0)}$ inside the identity.

p.94, lines 2, 4 and 6: It should say $\sup_{t\in(..}$ inside the inequalities.

p.94, line -3: It should say $\sup_{s\in(..}$ inside the inequality.

p.94, line 3: It should say $\sup_{s\in(..}$ inside the inequality.

p.96: Delete everything from line 6 (including Lemma 5.9 which was needed for the proof of Theorem 5.7 the latter being no longer required) to line 7 of page 98.

p.99, line 6: Should be $d =$ ... rather than $d \leq$ ....

p.99, line 10: d_0 instead of d

Delete the following text as it is the part of the proof of Theorem 5.4 which was based on Theorem 5.7 (which has now been omitted): p.103 line -6 to p.104 line 13.

p.106, line 1: ρ_1 instead of ρ_0

Now delete everything after 'We may therefore estimate (line 1) until 'leads to the estimate...' (line 8) and replace it by the following text:

Define the Radon measure

$$\mu_\lambda(C) \equiv \int_{M^\lambda_{-1/2}\cap C} \phi_\lambda \, d\mathcal{H}^n.$$

Since $\phi_\lambda \leq 8$ for $\lambda \in (0, \rho_1/\sqrt{n})$ we estimate

$$\mu\lambda(B_R) \leq 8\,\mathcal{H}^n(M^\lambda_{-1/2} \cap B_R) \leq 8AR^n$$

for all $R \leq d/\lambda$ in view of (5.24). Since $\operatorname{spt}\phi_\lambda \subset B_{\lambda^{-1}d}$ we also have

$$\mu_\lambda(B_R) \leq 8\mathcal{H}^n(M^\lambda_{-1/2} \cap \operatorname{spt}\phi_\lambda) \leq 8\mathcal{H}^n(M^\lambda_{-1/2} \cap B_{\lambda^{-1}d}) \leq 8A\,(d/\lambda)^n \leq 8AR^n$$

for all $R \geq d/\lambda$. Hence

$$\mu_\lambda(B_R) \leq 8AR^n$$

for all $R > 0$ and $0 < \lambda < \rho_1/\sqrt{n}$.

Lemma C.3 applied to μ_λ therefore implies that

$$\int_{M^\lambda_{-1/2}} \gamma_n \phi_\lambda \, d\mathcal{H}^n < \infty$$

and in particular leads to the estimate

$$E_2 \ldots$$

(Continue as before from here.)

p.106, line 13: ρ_1 instead of ρ_0

p.106, line - 5: ρ_1 instead of ρ_0

p.107, line 2: Insert 'depending only on n and A after 'large'.

p.112, line 10; $+$ instead of $-$

p.113, line 8: The last term in (A.6) should be R^M_{mikl}.

p.114, line 9: Replace this identity by $\partial_i \nu = A_i^j \partial_j F$.

p.114, line 13: The second term in the chain of identities should be replaced by $\partial_i \left(A_i^j \partial_j F \right)$.

p.114, line 13: insert between 'using' and 'the Codazzi' the phrase '$\partial_i g^{jk} = 0$ at our point of calculation (which is a consequence of the identity $\left(\partial_i \partial_j F\right)^T = 0$ above) to obtain the third identity and'

p.126: In the first displayed formula on this page replace $\mathbb{R}^{n+1}$ by $\mathbb{R}^N$ everywhere.

p.134, line -6: σ_j instead of σ

p.134, line -1: Should be $B_{\sigma_j/2}(y_j)$

p.136: Delete Theorem D.8 and Lemma D.9. including their proofs that is all of pages 136 and 137. The proof of Theorem D.8 presented here is incorrect for a similar reason the proof of Theorem 5.7 is wrong.

p.142, line 11: $\mathbb{R}^3$ instead of $\mathbb{R}^{n+1}$.

p.142, line 11: 'There exist absolute constants $\epsilon > 0$ and $c > 0$ such that ...'

REFERENCES

[B] K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Math. Notes Princeton, NJ, Princeton University Press 1978

[E1] K. Ecker *Regularity Theory for Mean Curvature Flow*, Birkhäuser 2004

[E2] K. Ecker, Partial regularity at the first time for hypersurfaces evolving by mean curvature, Mathematische Annalen September 2012, online publication DOI 10.1007/s00208-012-0853-6