HEAT EQUATIONS IN GEOMETRY AND TOPOLOGY

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1. Introduction

Non-linear heat equations have played an important role in differential geometry and topology over the last decades. Generally speaking, a geometric quantity or structure on a manifold is evolved in a canonical way towards an optimal one.

Examples are the harmonic map flow due to Eells and Sampson [ES] which finds *harmonic* maps, that is local minima of the energy functional. The *curve-shortening flow* [GH], [Gr] and its higher dimensional analogue, the *mean curvature flow* [Hu1], deform a curve (hypersurface) in the direction of its normal vector at every point, with speed equal to the curvature (mean curvature) at that point.

In 1982, Hamilton [Ha1] introduced the *Ricci flow* which deforms an initial metric in the direction of its Ricci tensor. This flow tends to improve the manifold to one with locally homogeneous geometry (as will be explained later). Hamilton used this to prove a number of topological classification results for manifolds with conditions on their curvature, such as for instance closed 3-manifolds with positive Ricci curvature and closed 4-manifolds with positive curvature operator (see [CLN] for a description of his work and a complete list of references). In 2003, Perelman [P1], [P2], [P3] completed Hamilton's Ricci flow programme [Ha3] which had the aim of settling Thurston's geometrization conjecture for closed 3-manifolds [Th1]. This conjecture had predicted such manifolds to be decomposable into pieces with locally homogeneous geometry.

In this article, we will present some of the main ideas starting in Section 2 with linear heat diffusion on a closed manifold and its interaction with the underlying geometry. In Section 3, we will show how the curve-shortening-flow can be used to prove the isoperimetric inequality in the plane. Section 4 focusses on a heat flow proof of the uniformization theorem for closed surfaces. Section 5 explains the Poincaré conjecture [Po] and Thurston's more general geometrization conjecture as well as Hamilton's programme for settling them via the Ricci flow. In Section 6, we present one of Perelman's main ideas leading to the final resolution of these conjectures.

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2. Heat diffusion on closed manifolds

We consider a smooth closed (compact and without boundary) n-dimensional Riemannian manifold (M, g). The metric g is given by a (smoothly varying) symmetric, positive definite $n \times n$-matrix (g_{ij}) at every point of M.

Let $f_0 : M \to \mathbb{R}$ be a smooth function. For the purpose of this exposition, one should think of f_0 as a temperature distribution on M. We now evolve f_0 by the *heat equation* which means that we consider a time-dependent function $f : M \times (0, T) \to \mathbb{R}$ which solves the partial differential equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) f = 0$$

with initial condition $f(0) \equiv f(\cdot, 0) = f_0 : M \to \mathbb{R}$. Again, one should think of $f(t) \equiv f(\cdot, t)$ as the temperature distribution on M at time t.

The Laplace-Beltrami operator on (M, g) is defined by

$$\Delta f = \nabla^i \nabla_i f = g^{ij} \left(\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f\right)$$

where (g^{ij}) denotes the inverse of the metric $g = (g_{ij})$, ∂_i stands for coordinate derivatives and ∇_j for covariant (coordinate system independent) derivatives of vector fields, in this case of the gradient vector field ∇f defined by $\nabla^i f \equiv g^{ij} \partial_j f$. The covariant derivatives involve both the ∂_i and the Christoffel symbols Γ_{ij}^k. We sum over repeated indices. If $M = \mathbb{R}^n$ with the standard metric this reduces to the well-known Laplace operator given by

$$\Delta f = \sum_{i=1}^{n} \partial_i \partial_i f.$$

In this paper, we shall need the *integration by parts formula* for manifolds without boundary

$$-\int_M \langle \nabla h, \nabla f \rangle \, d\mu = \int_M h \Delta f \, d\mu \tag{1}$$

of which the identity

$$\int_M \Delta f \, d\mu = 0 \tag{2}$$

is a special case. Here $d\mu$ denotes the volume element of (M, g) and the bracket $\langle \cdot, \cdot \rangle$ expresses the metric acting on vector fields.

The *average* of $f(t)$ over M (describing the average temperature of M at time t) is defined by

$$\bar{f}(t) = \frac{1}{vol(M)} \int_M f(t) \, d\mu.$$

Our first observation is that this average remains unchanged under the heat equation. Indeed, one calculates

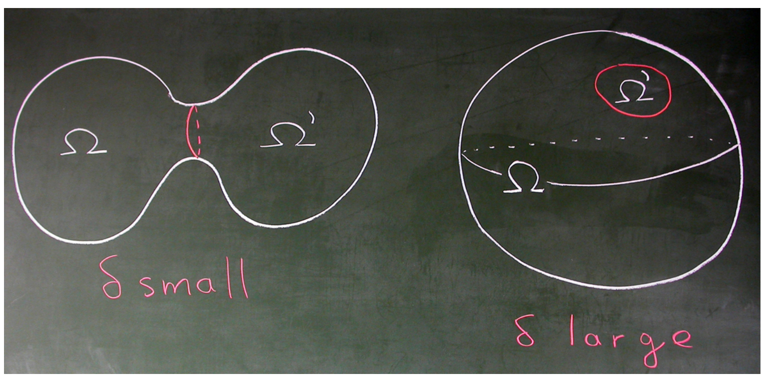
$$\frac{d}{dt} \bar{f}(t) = \frac{1}{vol(M)} \int_M \frac{\partial f}{\partial t}(t) \, d\mu = \frac{1}{vol(M)} \int_M \Delta f(t) \, d\mu = 0.$$

The last identity follows from (2). We therefore conclude that $\bar{f}(t) = \bar{f}(0) = \bar{f}_0$ for all $t \geq 0$.

We shall now see that heat diffusion takes place for $t \to \infty$ in the sense that over a large time period the temperature almost evens out. More precisely, $f(t) \to \bar{f}_0$ for $t \to \infty$ in the sense of smooth convergence. Here, we only establish the convergence in an integral sense. This can be derived reasonably easily from a geometric property of the manifold (M, g), the validity of the Poincaré inequality. The latter states that there is a constant $\delta > 0$ which only depends on (M, g) such that

$$\delta \int_M (u - \bar{u})^2 \, d\mu \leq \int_M |\nabla u|^2 \, d\mu$$

holds for all $u \in C^1(M)$. The geometric interpretation of this inequality is displayed in the following figure.



On a surface M, the constant δ must be small if two large regions, Ω and Ω', can be separated by a short loop as depicted in the left drawing. In the right picture, every small loop can only separate a small region from a large one. In particular, one can find a region for which the boundary length and enclosed area are 'in proportion to each other'. There are many examples of non-compact manifolds which do not admit a Poincaré inequality (such as for example the catenoid minimal surface).

Consider now the expression

$$\int_M (f(t) - \bar{f}_0)^2 \, d\mu$$

which measures the average deviation of the temperature at time t from the constant average temperature $\bar{f}_0$. We want to show, using the geometric information contained in the Poincaré inequality, that this integral tends to zero in the infinite future. To this effect, we calculate

$$\frac{d}{dt} \int_M (f(t) - \bar{f}_0)^2 \, d\mu = 2 \int_M (f(t) - \bar{f}_0) \frac{\partial f}{\partial t}(t) \, d\mu.$$

Inserting the heat equation and integrating by parts leads to

$$2\int_M (f(t)-\bar f_0)\,\Delta f(t)\,d\mu = -2\int_M \langle \nabla(f(t)-\bar f_0), \nabla f(t)\rangle\,d\mu$$

on the right hand side. Since $\bar f_0$ is a constant we obtain the identity

$$\frac{d}{dt}\int_M (f(t)-\bar f_0)^2\,d\mu = -2\int_M |\nabla f(t)|^2\,d\mu.$$

The Poincaré inequality applied to $u = f(t)$ now yields

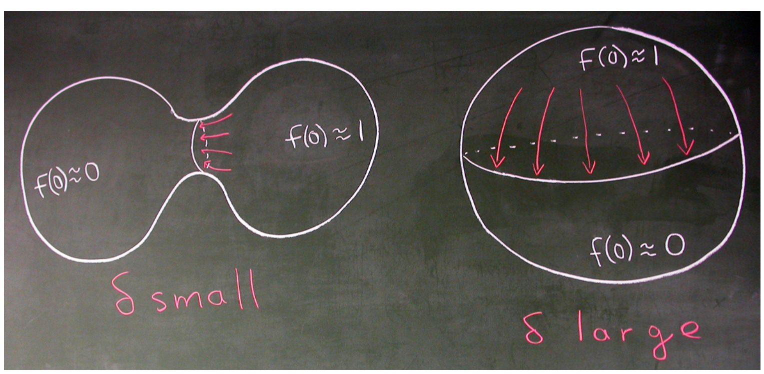
$$\frac{d}{dt}\int_M (f(t)-\bar f_0)^2\,d\mu \le -2\delta\int_M (f(t)-\bar f_0)^2\,d\mu$$

where we replaced $\bar f(t)$ by $\bar f_0$. Integrating, we finally conclude an exponential decay rate in time, that is

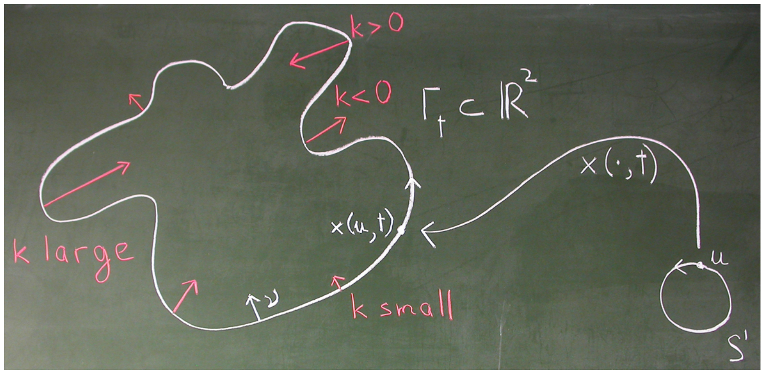
$$\int_M (f(t)-\bar f_0)^2\,d\mu \le e^{-2\delta t}\int_M (f_0-\bar f_0)^2\,d\mu \to 0$$

for $t\to\infty$.

The following figure suggests that the smaller δ is (such as on closed manifolds with tight necks), the more slowly diffusion of heat takes place. In this sense, optimal diffusion happens on spheres, albeit still in infinite time (at least for mathematicians). In fact, we could have defined the optimal δ in the Poincaré inequality solely in terms of the diffusion rate for the heat equation on M.



We could have also used Fourier series to derive this result. The optimal δ in the Poincaré inequality is then proportional to the lowest Fourier mode which in turn is given in terms of the smallest eigenvalue of the negative Laplace - Beltrami operator on M (with respect to our sign convention).



The figure displays one member of a family of closed, embedded curves Γ_t given by maps $x(\cdot,t) : S^1 \to \mathbb{R}^2$ which evolve by the law of motion

$$\frac{\partial x}{\partial t} = k\nu \tag{3}$$

called *curve-shortening flow*. Here ν is the inward pointing normal, k the curvature function of Γ_t and x is short for $x(u,t)$. We can express the equation also in terms of the arc length parameter $s = s(u,t)$ of the evolving curve. Using one of the definitions of curvature, the equation then reads as

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}.$$

This looks deceptively like a linear heat equation for x (actually a system of two equations in our case), however the non-linearity is hidden inside s which depends on products of the spatial derivatives of x. If, for instance, we express the evolving curve locally in terms of a function $h : (a,b) \times (0,T) \to \mathbb{R}$ with variables z and t then the equation

$$h_t = \frac{h_{zz}}{1 + h_z^2}$$

results.

From equation (3), the evolution of any other geometric quantity on the curve can be computed [GH]. The curvature, for instance, satisfies

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 \tag{4}$$

which is a so-called *reaction-diffusion equation*.

If we only considered the diffusion part

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2},$$

the curvature function $k(t)$ on Γ_t would tend to a constant for $t \to \infty$ as we have seen earlier for our temperature f. The curve would slowly turn into a circle. The reaction part of the equation,

$$\frac{\partial k}{\partial t} = k^3,$$

with positive $k(0)$, has the explicit solution

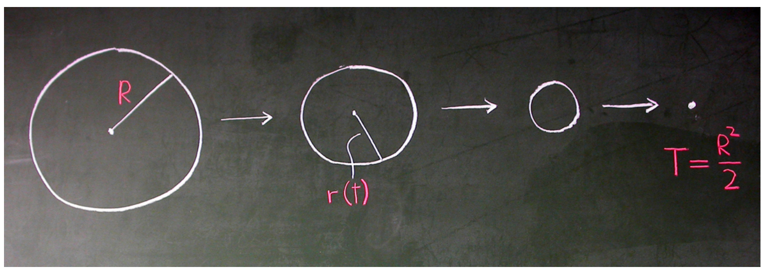
$$k(t) = \frac{1}{\sqrt{\frac{1}{k(0)^2} - 2t}}$$

so that the *blow-up* time is given by

$$T = \frac{1}{2k(0)^2}.$$

In the equation (4) for the curvature function of Γ_t, these two effects are competing. It is an extremely difficult analytic problem to understand this interaction for a general initial curve. Fortunately, the geometric nature of the equation comes to the aid of the analysis (see [GH], [Gr], [Hu2]).

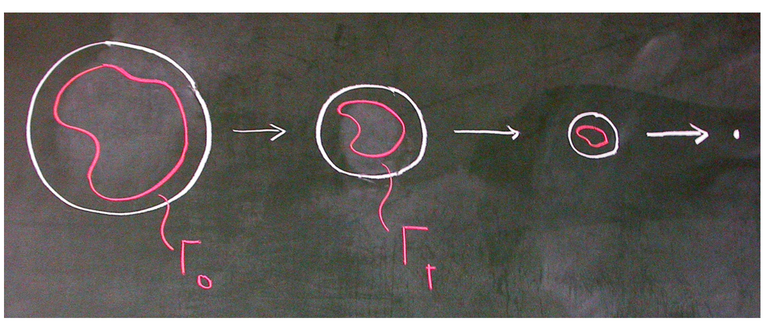
An example of a curve exhibiting the 'reaction behaviour' is the round circle.



The radius of the circle at time t is $r(t) = \sqrt{R^2 - 2t}$ which vanishes at time $T = R^2/2$.

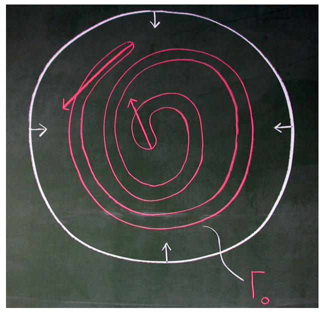
One of the most important features of the curve-shortening flow which it shares (after appropriate reformulation) with many other heat type equations is the

Comparison principle *Initially disjoint embedded closed curves stay disjoint during the curve-shortening flow.*

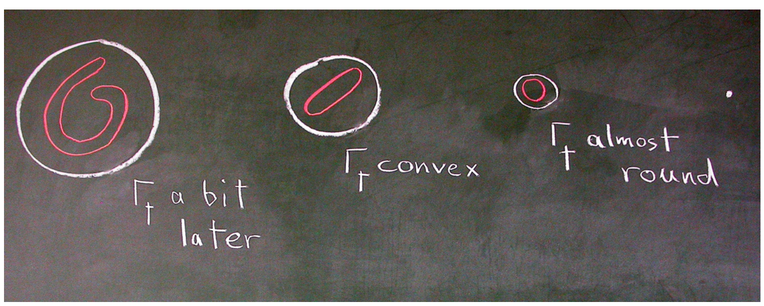


The picture shows that, by comparison with a shrinking circle, every closed curve can only exist for a finite time. This is one of the reaction effects.

Consider now the following initial situation (borrowed from a beautiful survey article by Brian White [Wh]):

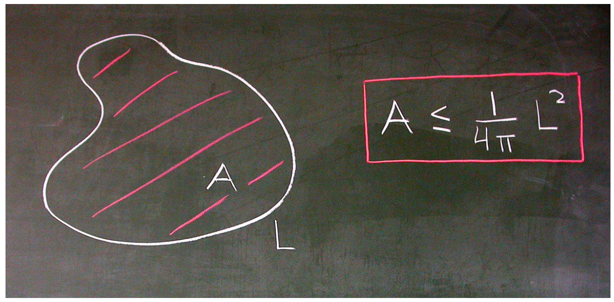


Any enclosing circle will squash the spiral-like curve down to a point in finite time. On the other hand, the high curvature at the tips is expected to uncoil the curve (which one should consider as a diffusion effect). The movie shows several stages of what happens.

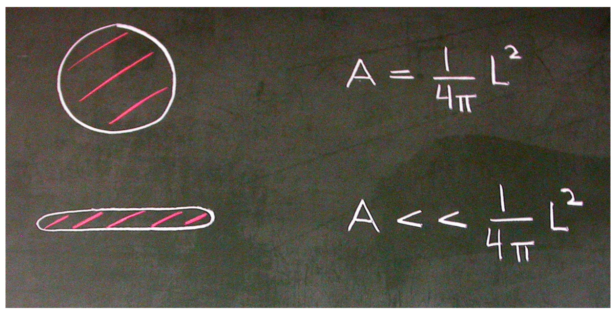


In fact, it follows from Grayson's work in 1987 [Gr], (see [Hu2] for an easier proof) that the curve will stay smooth and embedded and will become convex before its extinction time. Then, by a result of Gage and Hamilton in 1984 [GH], it will become asymptotically round in a smooth fashion. More precisely, after rescaling the evolving curve so as to keep for instance the enclosed area fixed (which results in a slightly different flow), it converges smoothly to a round circle. This is a consequence of the diffusion term: The diffusion cannot stop the formation of a singularity but it is strong enough to preserve embeddedness of the curve and produce a very 'symmetric singularity'.

We will now show how a geometric property of the plane (which is actually related to, albeit stronger than, the Poincaré inequality) can be derived using curve-shortening flow. This property is the famous *isoperimetric inequality* stated below.



Extreme cases are circular disks for which equality holds and regions with tiny areas surrounded by long curves.



The isoperimetric inequality can also be viewed as a

Variational principle *Among all open and bounded subsets of the plane with equal perimeter, circular disks have the largest area.*

The word 'perimeter' here refers to the measure for a very general class of boundary contours which of course includes all smooth ones. An elementary and very elegant outline of the proof can be found in [HT].

In this article, however, we would like to derive the inequality using the curve-shortening flow (this argument was first brought to our attention by Ben Andrews, see also [To]). We start with an arbitrary initial curve $\Gamma = \Gamma_0$ with enclosed area $A = A(0)$ and length $L = L(0)$. Using the formulas for the change of length and area

$$\frac{d}{dt}L(t) = -\int_{\Gamma_t} k^2\,ds \qquad (< 0)$$

(which justifies the term *curve-shortening*) and

$$\frac{d}{dt}A(t) = -\int_{\Gamma_t} k\,ds$$

we infer that

$$\frac{d}{dt}\left(L(t)^2 - 4\pi A(t)\right) = -2L(t)\int_{\Gamma_t} k^2\,ds + 4\pi\int_{\Gamma_t} k\,ds.$$

Since our curve is closed and embedded, the winding number theorem implies for all t

$$\int_{\Gamma_t} k\,ds = 2\pi,$$

so that, also using the Cauchy-Schwarz inequality for integrals, we can estimate the second term by

$$4\pi \int_{\Gamma_t} k\,ds = 2\left(\int_{\Gamma_t} k\,ds\right)^2 \leq 2L(t)\int_{\Gamma_t} k^2\,ds.$$

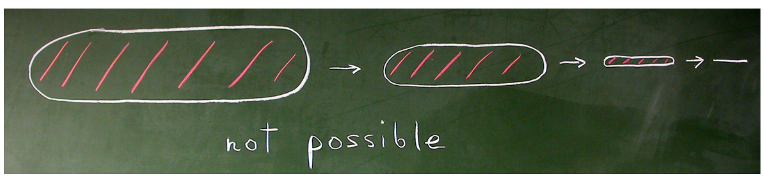
Hence we arrive at the relation

$$\frac{d}{dt}\big(L(t)^2 - 4\pi A(t)\big) \leq 0.$$

Gage-Hamilton's and Grayson's theorems imply

$$L(t)^2 - 4\pi A(t) \to 0$$

for $t \to T < \infty$. The most difficult step in proving this statement consists in showing that area and length vanish *at the same time*. In particular, the following evolution is not possible under curve-shortening.



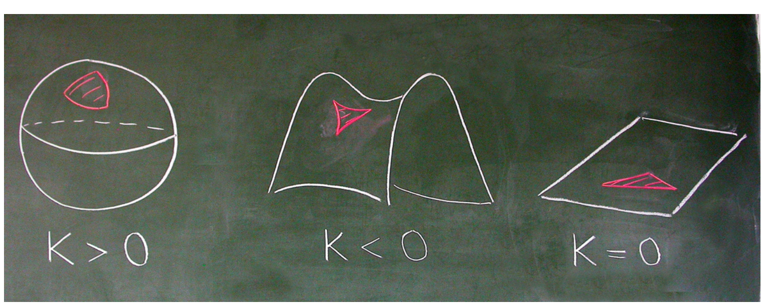
Because our quantity is also non-increasing we conclude that it must have initially been non-negative, that is

$$A(0) \leq \frac{1}{4\pi} L(0)^2.$$

Since $\Gamma = \Gamma_0$ and hence $A = A(0)$ and $L = L(0)$ were arbitrary, this proves the isoperimetric inequality for all smooth, closed and embedded curves.

4. Geometric classification of surfaces

The picture below shows surfaces of positive, negative and zero curvature.

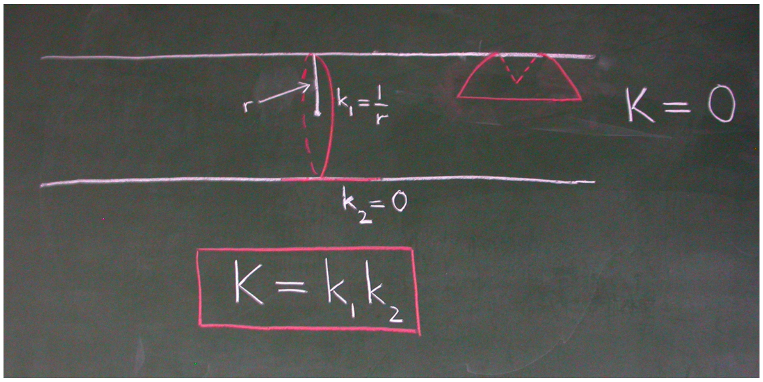


In the case where the curvatures are one, negative one or zero, the corresponding geometries (metrics) are called *spherical*, *hyperbolic* and *Euclidean* (or *flat*).

The term curvature here refers to the Gauß curvature, an intrinsic quantity, that is one which can be computed from within the surface, for example by measuring the deviation of the angle sum of geodesic triangles from π. On positive curvature surfaces such as the sphere, the angle sum within a triangle with geodesic segments (great circle arcs on S^2) as sides is larger than π, on a saddle surface less than π. The (appropriately defined) 'infinitesimal' angle sum deviation from π at a point is called the Gauß curvature of the surface at this point.

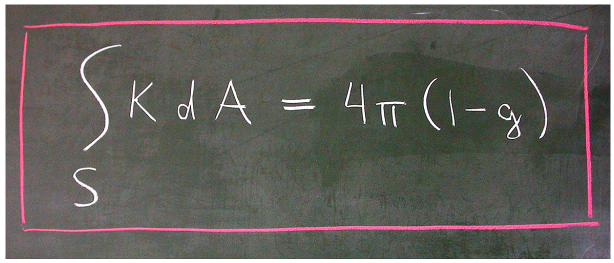
For a surface S embedded in $\mathbb{R}^3$, Gauß' famous *theorema egregium* states that his curvature at a given point p on S can be alternatively defined as the product of the principal curvatures at p. To compute the principal curvatures at p one first considers the curve segment within S obtained by intersecting S with a plane containing a normal vector to S at p. One computes the curvature of this curve segment at p (in the usual way) and then varies over all directions of planes normal to S through p. The minimum and maximum curvatures thus obtained are called *principal curvatures* at p. The sign of these depends on the choice of unit normal to S, so may vary by a factor of -1.

On S_r^2, the sphere of radius r, both principal curvatures equal $1/r$ everywhere, since they are the curvatures of great circle segments. Therefore, S_r^2 has Gauß curvature equal to $1/r^2$. On a surface with $K < 0$, the two principal curvatures must have opposite signs everywhere. The surface therefore looks saddle-shaped near every point.



Note that the cylinder $S_r^1 \times \mathbb{R}$ depicted above can be obtained by rolling up a strip in $\mathbb{R}^2$. In the plane, all triangles have an angle sum equal to π and rolling up a piece of paper does not cause any distortions in the direction perpendicular to the rolling process. Therefore the angle sum of geodesic triangles on the cylinder is also equal to π as the above work of art attempts to convey. Therefore, the cylinder is (intrinsically) flat. As the reader can easily check, one of the principal curvatures equals zero as it occurs when we look in the direction of the cylinder axis. This proves Gauß' theorem (for cylinders).

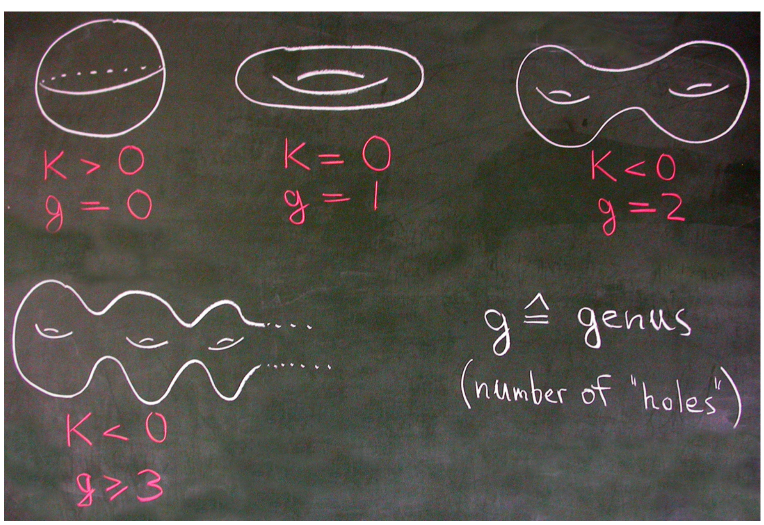
The Gauß-Bonnet theorem for a closed surface S of genus g (number of handles or 'holes') states that



This theorem is the prime example of a link between geometric and topological properties of surfaces. In particular, it gives obstructions to the existence of certain geometries on certain surfaces.

For instance, a torus can neither admit a geometric structure (metric) with everywhere positive curvature (spherical geometry) nor one with everywhere negative curvature (hyperbolic geometry), while surfaces with more than one handle could admit hyperbolic geometry (the latter is initially a bit hard to visualize; we refer the reader to the book by Weeks [We] where hyperbolic structures on surfaces of genus greater than or equal to two are constructed in an intuitive way). That a torus may admit flat geometry is easier to see by first rolling up a square to form a piece of cylinder in $\mathbb{R}^3$ and then rolling up this cylinder piece in a fourth direction (which ensures that the angle sum in triangles does not change). The result is an embedding of a flat torus in $\mathbb{R}^4$.

There is a topological classification of closed (compact and without boundary) orientable surfaces S via their genus g (see [We] for a lucid exposition).



The *uniformization theorem* states that all closed orientable surfaces admit metrics of Gauß curvature equal to either one, zero or negative one. Hence genus zero surfaces admit spherical geometry, tori flat geometry and all surfaces with more than one handle hyperbolic geometry.

In fact, the precise statement is even stronger in the sense that every given metric on a closed orientable surface is *conformal* to a metric of constant curvature (that is a constant curvature metric multiplied by a smooth positive function on the surface) . This was first proved using complex analytic methods.

Hamilton [Ha2] (and Chow [Ch] for part of the positive curvature case) established a canonical method for finding constant curvature metrics in a given conformal class. Their approach still uses the uniformization theorem at some stage (in the positive curvature case) but the necessity for this has been removed in more recent work, see [CLT].

Hamilton employed heat diffusion in the form of his *volume normalized Ricci flow* for surfaces [Ha2]: Here a family of metrics $(g(t))$ on a surface S (symmetric, positive 2×2-matrices at every point of S) evolves by the equation

$$\frac{\partial g}{\partial t} = -(K - \bar{K})g$$

where

$$\bar{K} = \frac{1}{Area_t(S)} \int_S K \, dA_t = \frac{4\pi(1-g)}{Area_t(S)}$$

is the average curvature of S and dA_t is the area element of $(S, g(t))$.

It can easily be shown that this flow keeps the area of S fixed so that by the Gauß-Bonnet formula the average curvature also does not change. Since constant curvature metrics obviously remain unchanged, this process is therefore a good candidate for the task of deforming an arbitrary initial metric into one with constant curvature.

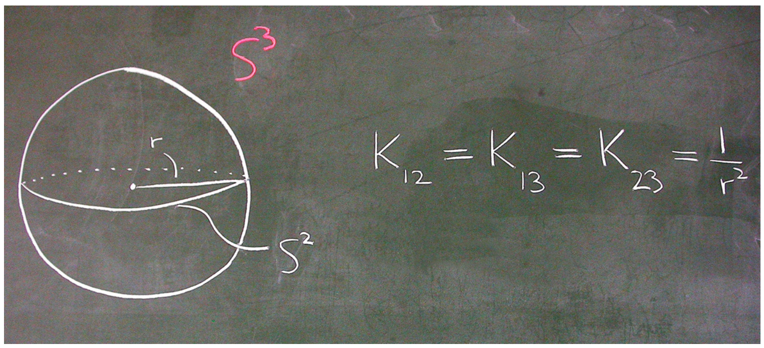
If we start with an initial metric $g(0)$ and consider only metrics conformal to it, that is of the form

$$g(t) = e^{u(t)} g(0)$$

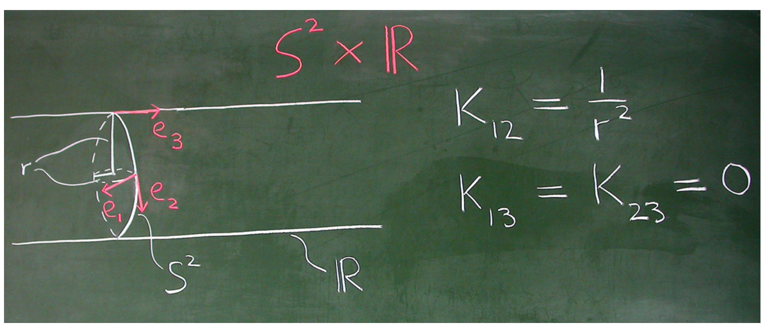
for some smooth function $u(\cdot, t)$ on S, then u has to satisfy a certain non-linear heat equation which we will not spell out here. Using this differential equation, Hamilton [Ha2] and Chow [Ch] (in the positive curvature case) showed that $g(t)$ converges smoothly as $t \to \infty$ to a limiting metric $g(\infty) = e^{u_\infty} g(0)$ with constant curvature.

5. Ricci flow and the geometric classification of 3-manifolds

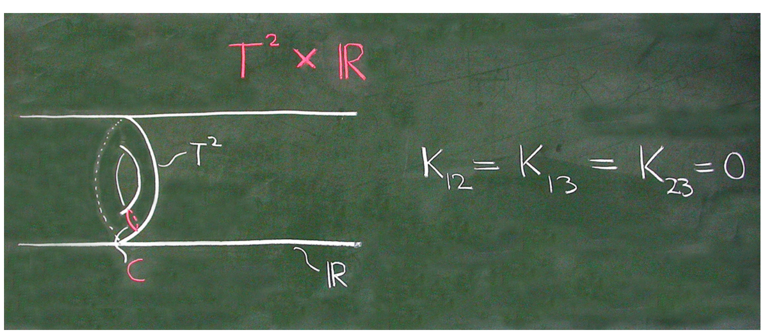
On a three-dimensional Riemannian manifold (M, g) one can think of sectional curvatures roughly in the following way: At a point p in M we consider a plane Π inside the tangent space of M at p. One now locally chooses a suitable surface S in M containing p with $T_pS = \Pi$, somewhat like a 2-dimensional local cross-section of M near p. The sectional curvature of M at p with respect to Π is defined as the Gauß curvature of S at p. For an orthonormal basis e_1, e_2, e_3 of T_pM on a 3-manifold there are three canonical choices of planes. We denote the sectional curvatures with respect to these by K_{12}, K_{13} and K_{23}. A 3-manifold for which the three sectional curvatures are independent of the particular point is called *homogeneous*. If in addition the curvatures are also independent of the plane directions at each point, we call it *isotropic*. Below, we show some examples.



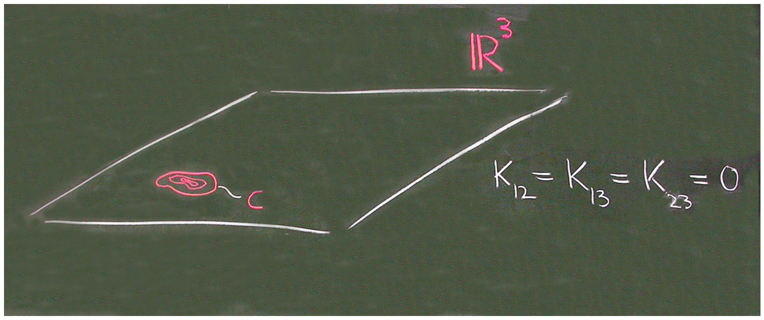
S^3 is *isotropic* so in particular *homogeneous* and $S^2 \times \mathbb{R}$ is *homogeneous* but not *isotropic*.



The manifold $T^2 \times \mathbb{R}$ is flat but not simply connected, that is a loop through the hole of the torus cannot be contracted to a point inside M,



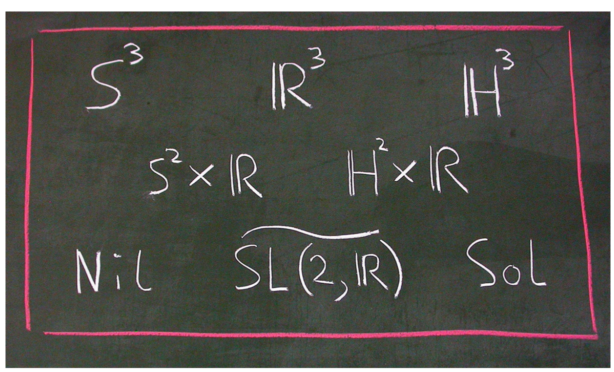
but Euclidean space $\mathbb{R}^3$



is flat and simply connected.

There are three homogeneous spaces which serve as geometric models for surfaces: the sphere, hyperbolic space and Euclidean space. In three dimensions, their analogues $S^3, \mathbb{H}^3$ and $\mathbb{R}^3$ are not the only possibilities as we have already seen above.

Thurston formulated some natural conditions, apart from homogeneity, which a three-dimensional model geometry should satisfy. Among these are simple connectedness, maximal symmetry and the existence of at least one compact manifold modelled by it. For example the torus T^3 which is compact carries the same geometry as $\mathbb{R}^3$. Another condition requires that the symmetry group has a compact point stabilizer which in two dimensions would rule out surfaces with only helicoidal symmetry. Thurston came up with a complete list of eight such geometries which, as he conjectured, would form the building blocks for all closed (compact without boundary) orientable 3-manifolds:



The first five of these are self-explanatory. The 3-manifolds listed in the last row are so-called unimodular Lie groups (we refer the reader to the excellent book by Thurston [Th2] for details). Roughly speaking, *Nil*, also called the Heisenberg group, is a twisted version of $\mathbb{R}^2 \times \mathbb{R}$. The manifold $\widetilde{SL(2,\mathbb{R})}$ is the universal cover of the symmetry group of $\mathbb{H}^2$ which one can think of as a twisted version of $\mathbb{H}^2 \times \mathbb{R}$. The last 3-manifold is a particular type of torus bundle over the circle (but not as simple as $T^2 \times S^1$; there must be a certain distortion of the geometry of the T^2

fibres, see [Th2] for details and [We] for additional intuitive descriptions).

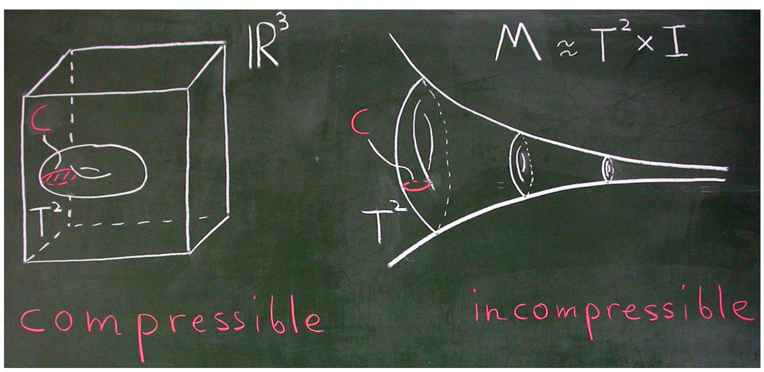
Geometrization conjecture ([Th1]) *Every closed, orientable* 3*-manifold can be decomposed along* 2*-spheres and incompressible* 2*-tori and then capped off along the* 2*-spheres by* 3*-balls in such a way that the resulting finitely many pieces each admit one of the above eight geometries.*

More precisely, they locally look like one of these eight, possibly up to discrete group actions. For instance, $S^2 \times S^1$ is modelled by $S^2 \times \mathbb{R}$ and T^3 by $\mathbb{R}^3$.

The above manifolds are all non-compact, except S^3. All are simply connected. After some additional thought, one concludes that the geometrization conjecture implies the famous

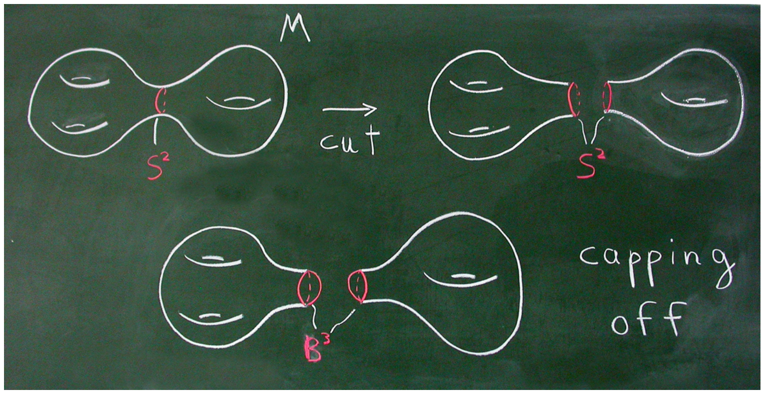
Poincaré conjecture ([Po]) *Every closed and simply connected* 3*-manifold is homeomorphic to* S^3.

The following picture gives an intuitive idea of the term *incompressible torus.*



In the left picture, the circle through the hole of the torus can be contracted in $\mathbb{R}^3$ but not on T^2. In the right picture, the torus in some sense 'represents' the topology of M, so the circle cannot be contracted, neither on T^2 nor in M.

The terms *decomposing* or *'cutting'* and *'capping off'* are depicted below. The two operations combined are often referred to as *surgery*, the reverse process is called *forming a connected sum.*

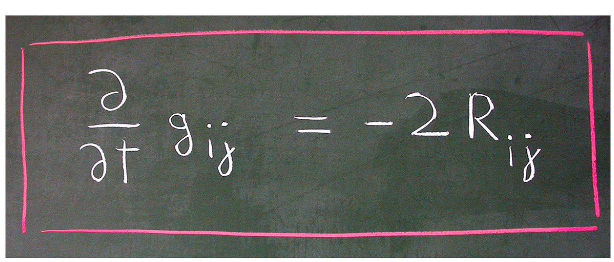


Thurston himself [Th3],[Th4],[Th5] contributed to the resolution of his conjecture by classifying 3-manifolds satisfying additional conditions which included all hyperbolic manifolds. However, the final resolution was based on results of Hamilton's **Ricci flow programme** [Ha3] which was completed around five years ago in the work of Perelman [P1], [P2] and [P3].

Let us explain some of the basic ideas of this programme. The three *Ricci curvatures* of a 3-manifold are averages of the sectional curvatures in the directions of the vectors of a 3 - frame $\{e_1, e_2, e_3\}$ as follows:

$$\begin{aligned} R_{11} &= Ric(e_1, e_1) = K_{12} + K_{13} \\ R_{22} &= Ric(e_2, e_2) = K_{12} + K_{23} \\ R_{33} &= Ric(e_3, e_3) = K_{13} + K_{23} \end{aligned}$$

Of course one can analogously define sectional and Ricci curvatures in higher dimensions but there the Ricci curvatures, being averages, contain less geometric information than the sectional curvatures. It is a special feature of three dimensions that the R_{ii} completely determine the K_{ij} as one can easily see from the definitions. One should think of the *Ricci tensor* (R_{ij}) as a symmetric 3×3-matrix at every point of the manifold whose eigenvalues are the Ricci curvatures. Of course, the metric $g = (g_{ij})$ on a 3-manifold is also given by a 3×3-matrix field which is symmetric and in addition positive definite. Hamilton's Ricci flow for a 'time-dependent' family of metrics $g(t)$ on a 3-manifold M is defined by the evolution law



where we start with some metric $g(0) = g_0$. The definition also makes sense in arbitrary dimensions. In two dimensions, there is only one sectional curvature. Therefore the Ricci flow reduces to the equation considered in the previous section on surfaces (without the average curvature term).

In suitable (so-called *harmonic*) coordinates for the manifold, the Ricci tensor looks like

$$-2R_{ij} = \Delta g_{ij} + O(\partial g_{ij}).$$

The Laplacian here acts on the components of the metric tensor and the last term is non-linear in the coordinate derivatives of the metric. This immediately suggests that the Ricci flow system is a type of non-linear heat equation for the metric. It turns out that the equation for the metric is not as frequently used (except in the proof of short-time existence of a solution of Ricci flow for given initial metrics) as the evolution equation for the *curvature operator* $\mathcal{R}$ of the metric. The curvature operator in three dimensions is a symmetric 2-tensor whose eigenvalues are the sectional curvatures. It satisfies the equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{R} = \mathcal{R}^2 + \mathcal{R}^{\#}.$$

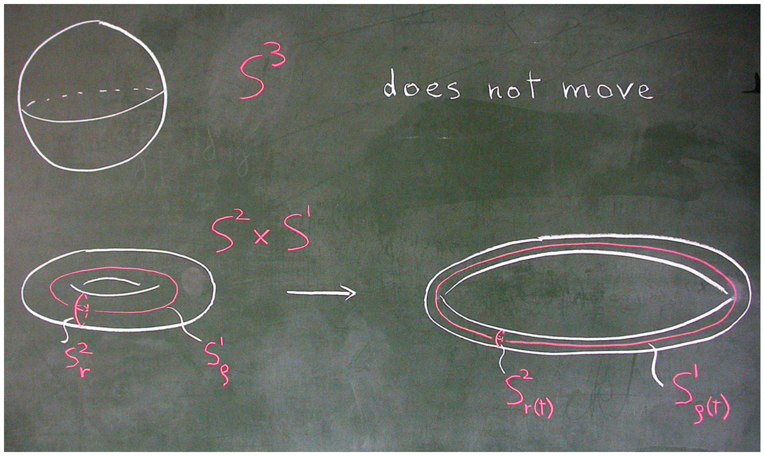
Let us abbreviate the sectional curvatures by $\lambda \leq \mu \leq \nu$ (the order is preserved by the Ricci flow). The tensor $\mathcal{R}^{\#}$ which is symmetric and quadratic in $\mathcal{R}$ is given in diagonalized form by its eigenvalues $\mu\nu, \lambda\nu, \mu\nu$. The curvature therefore satisfies a reaction-diffusion equation just as for the curve-shortening flow except this one has a quadratic rather than a cubic non-linearity which is due to the fact that intrinsic curvature scales like the product of extrinsic curvatures (see the theorema egregium).

The reaction term can cause singularities in finite time for certain initial metrics. For example, a round S^3 and some 3-manifolds close to it 'contract' in finite time T. In this case, the curvature explodes like $(T-t)^{-1}$. To understand what is happening asymptotically to the geometry one considers instead the *volume normalized Ricci flow* for manifolds with finite volume (such as closed ones) . In three dimensions, this flow is defined by

$$\frac{\partial}{\partial t} g_{ij} = -2\left(R_{ij} - \frac{\bar{R}}{3} g_{ij}\right)$$

where $\bar{R}$ denotes the average of the scalar curvature function given by $R = R_{11} + R_{22} + R_{33}$. In the case of surfaces, the flow reduces to the one discussed in the previous section. It keeps the volume of the evolving manifold fixed such that for example S^3 and all other closed 3-manifolds carrying isotropic geometry (for instance T^3 or $T^2 \times S^1$) remain unchanged.

In fact, under the normalized flow, none of the closed quotients of the eight model geometries develop singularities except those for $S^2 \times \mathbb{R}$. For instance, $S^2 \times S^1$ expands to infinity in finite time while simultaneously thinning out in order to maintain its volume. The latter causes the curvatures to tend to infinity in finite time, as in the following figure.



For certain manifolds, the diffusive effect of the Ricci flow dominates after normalization. In fact, Hamilton's first breakthrough in 1982 which established the Ricci flow as a valuable tool for the study of the relation between geometry and topology of 3-manifolds was the following

Theorem (Hamilton 1982 [Ha1]**)** *If* (M^3, g_0) *is closed, orientable and simply connected and has positive Ricci curvatures then the solution* $(M^3, g(t))$ *of the normalized Ricci flow on* M^3 *which starts from* g_0 *smoothly converges to the standard metric on* S^3 *as* $t \to \infty$.

This implies in particular that any 3-manifold with positive Ricci curvatures has to be diffeomorphic to a sphere.

For a general initial metric, singularities may develop in finite time under the Ricci flow [Si], in infinite time after normalization, except for $S^2 \times S^1$ which we discussed above. Such singularities are detected by considering $|\mathcal{R}(t)|$, the square root of the sum of the squares of the sectional curvatures, which tends to infinity near the singularity as the singular time is approached.

To be able to study the solution near the singularity, one rescales the metric so as to keep the curvatures bounded, that is one considers

$$g_k(t) = \lambda_k \, g\left(\frac{t}{\lambda_k} + t_k\right)$$

where the λ_k are approximately the maxima of $|\mathcal{R}(t_k)|$ for a sequence of times t_k tending to the singular time, i.e. $\lambda_k \to \infty$. This increases distances but does not effect the Ricci curvatures which correspond to the speed. Rescaling also in time is therefore necessary because the solution takes longer to cover the larger distances.

If we also translate in time so as to set the singular time equal to zero, the rescaled solution will extend its existence further and further into the past. It has been a difficult problem to ensure that this rescaling process converges again to a solution of Ricci flow, at least for a subsequence of times. This problem was ultimately overcome by Perelman.

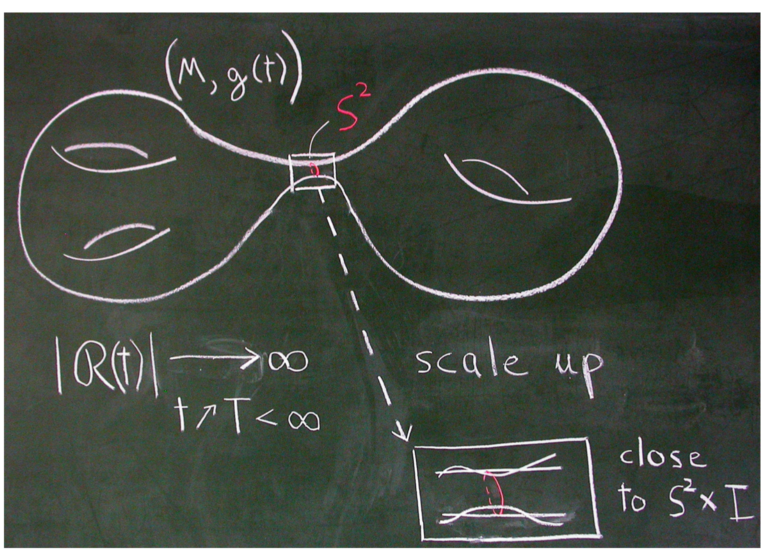
Since the rescaling limit has existed infinitely long (we call such a solution *ancient*), the diffusion term in the equation for the curvature operator has had enough time to improve the solution so we expect it to have nicer properties than the original one. Indeed, it follows from work of Hamilton [Ha3] and Ivey [I] that the lowest of the sectional curvatures becomes less and less negative during the rescaling process so that the rescaling limit has non-negative sectional curvatures. The latter follows by studying the dynamical behaviour of the system of ordinary differential equations

$$\frac{\partial}{\partial t}\mathcal{R} = \mathcal{R}^2 + \mathcal{R}^\#$$

which in diagonal form reads as

$$\frac{d}{dt}\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix} = \begin{pmatrix}\lambda^2+\mu\nu\\ \mu^2+\lambda\nu\\ \nu^2+\lambda\mu\end{pmatrix}.$$

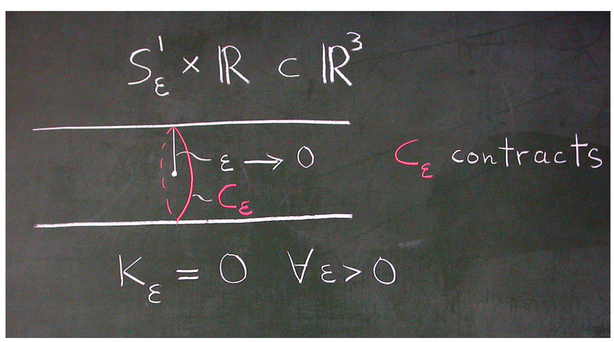
This information about the rescaling limit (in some sense this limit describes the structure of the original solution near the singularity) rules out a large number of possible geometries. The only remaining possibilities are S^3, $S^2\times\mathbb{R}$ as well as certain 'undesirable' structures which we will discuss later. In the case of S^3, we are essentially in the situation already covered by Hamilton in 1982, so we know what happens. If we see $S^2\times\mathbb{R}$ geometry we could for instance be sitting on a topological $S^2\times S^1$ contracting to a circle. Alternatively, our unscaled evolving metric could get closer and closer to a 'round neck' $S^2\times I$ near the singularity, where I is an interval, as in the picture below.



In this situation, a careful quantitative version of the surgery procedure of the kind explained earlier can be applied, that is the solution can be decomposed along an S^2 (as forecast in the geometrization conjecture) by removing an almost cylindrical piece and then capping off the resulting two pieces by two 3-balls. Here, the capping off has to be carried out carefully so that the resulting manifold has positive curvature near the caps which ensures that the two pieces initially 'move away from each other' in this region. In the case of the 'undesirable' rescaling limit, surgery as above is not possible. In fact, in this situation one cannot even make proper sense of the term 'rescaling limit'. This had been one of the main technical problems for two decades. Showing that for a closed (compact without boundary) solution this undesirable behaviour cannot occur after *finite time* was one of Perelman's contributions [P1]. We will explain this issue a little later.

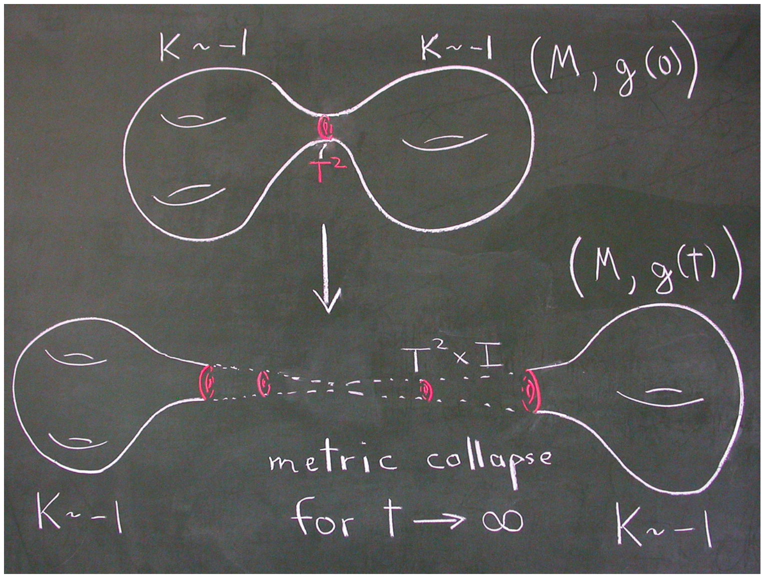
Hamilton [Ha4] also studied the behaviour of *non-singular* closed 3-manifold solutions of the *volume normalized Ricci flow* for time tending to infinity. By non-singular we mean that there is a curvature bound for the solution which is independent of time. This is something one might hope to obtain after no more singularities form. Hamilton showed that two of the possibilities could be smooth convergence to one of the isotropic (constant sectional curvature) geometries or metric collapse of the solution.

Metric collapse refers to a situation where on a family of manifolds with *uniformly bounded curvatures* (as our normalized Ricci flow solution) some shortest geodesics shrink away, as in the following picture of a collapsing sequence of cylinders in $\mathbb{R}^3$.



In this case, the curvatures are even zero. A more sophisticated (and three-dimensional) example is given by the Berger spheres (see [Th2]).

A typical example of the third possible behaviour one could observe for non-singular solutions as time tends to infinity is shown below (this example has been borrowed from [Ha3] where also further details can be found).



Under normalized Ricci flow the 'ends' of the manifold which are almost hyperbolic (that is K which describes the (almost identical) sectional curvatures is close to negative one) will hardly move while the 'middle' part which is almost Euclidean (with $T^2 \times I$ topology) stretches (as the volume has to stay constant) with resulting metric collapse.

The 'thick' parts at the ends will become hyperbolic in the limit. The geometric structure of the collapsing parts of the manifold is well-understood [CG1], [CG2]. They are so-called Seifert fibred spaces (which satisfy the prediction of the geometrization conjecture). The collapsing parts are connected to the hyperbolic ends by pieces which typically have the structure of $T^2 \times I$ that is correspond to the predicted incompressible tori. In effect, the Ricci flow provides a canonical way of finding a *thick-thin decomposition* of our manifold, as the latter is termed by 3-manifold topologists (see [Th2]).

Let us summarize **Hamilton's programme** for establishing the geometrization conjecture:

One starts with an arbitrary metric on a closed orientable 3-manifold. Under the evolution by the Ricci flow, it will develop singularities in finite time. If the volume of the manifold disappears at the first singular time, one obtains an S^3 or an $S^2 \times S^1$ after rescaling.

If not (and *assuming* that the 'undesirable' rescaling limits mentioned before do not occur in finite time) an $S^2 \times I$ neck will form on which one then performs surgery in the way described earlier. Letting the resulting pieces evolve separately, the procedure will repeat itself. If this happens only finitely often, for example if all surgery components vanish under the flow in finite time (*extinction time*), one ends up with a connected sum of finitely many S^3 quotients and $S^2 \times S^1$ components.

If the initial manifold was simply connected we obtain only S^3's and hence the Poincaré conjecture is confirmed. Perelman [P3] and Colding-Minicozzi [CM] could show that for simply connected 3-manifolds the extinction time is indeed finite.

If after finite time we are in the lucky situation of having a non-singular solution of the volume normalized Ricci flow, then the above three possibilities for non-singular solutions will occur, all of which confirm geometrization. Perelman [P2] could show that even if the formation of singularities persists forever, Hamilton's arguments could be suitably modified and augmented to confirm the geometrization conjecture.

6. Ricci flow and a Poincaré type inequality

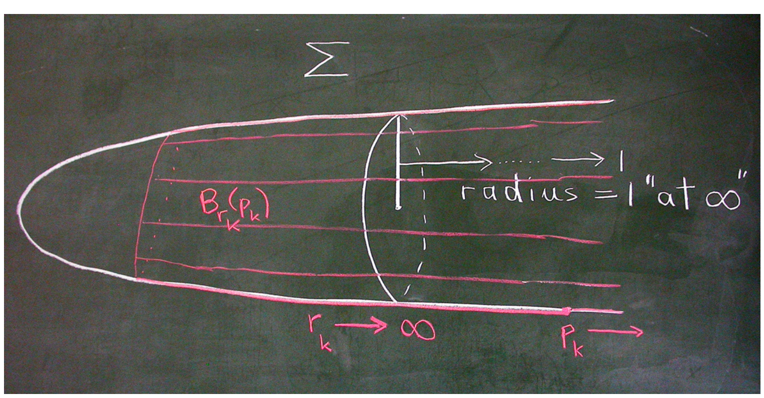
Let us last focus on one of Perelman's major contributions which was among the ingredients of the resolution of the geometrization conjecture.

Prior to Perelman's breakthrough, for many years the holy grail in Ricci flow theory had been to find a way of ruling out the 'undesirable' rescaling limit we have mentioned several times. Remember that rescaling limits describe the local geometry (and topology) of the Ricci flow solution near a singularity.

An example of such a solution is constructed as follows. In two dimensions, one considers the manifold $\Sigma = (\mathbb{R}^2, ds^2)$ where

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

This so-called *cigar soliton* solution of the two-dimensional Ricci flow found by Hamilton exists for all time (*eternal* solution). Furthermore, it evolves only by diffeomorphisms, that is points are moved around on the manifold but there is no change in its geometry. Embedded in $\mathbb{R}^3$ it would look like the picture below.

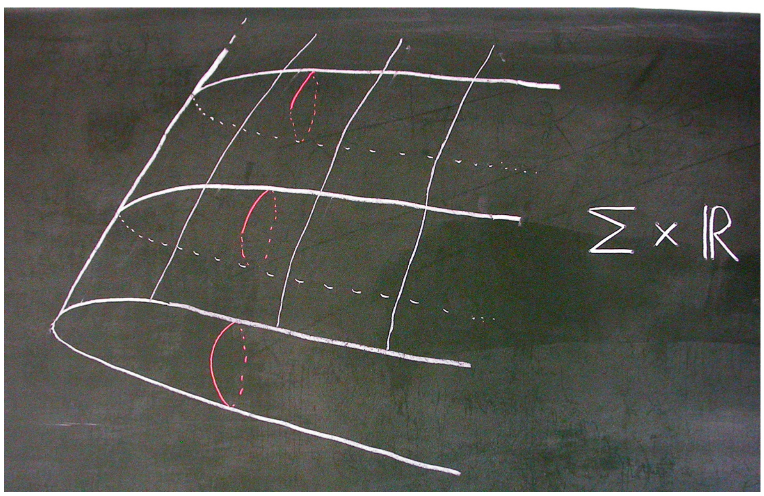


A special feature of this manifold is that as seen from infinity it looks like a half-line rather than a two-dimensional object in the sense that it dimensionally collapses at large radii in the following way:

$$\frac{Vol(B^2_{r_k}(p_k))}{r_k^2} \sim \frac{r_k}{r_k^2} \to 0$$

for a suitably chosen sequence of geodesic balls in Σ with radii r_k tending to infinity.

In three dimensions, we consider $\Sigma \times \mathbb{R}$



which for some sequence of geodesic balls satisfies

$$\frac{Vol(B^3_{r_k}(p_k))}{r_k^3} \sim \frac{r_k^2}{r_k^3} \to 0.$$

The picture suggests that as seen from infinity $\Sigma \times \mathbb{R}$ looks like a half-plane rather than a three-dimensional object. If our Ricci flow solution is geometrically close to $\Sigma \times \mathbb{R}$ near the singularity, then in small balls around the singularity it will deteriorate dimensionally. Perelman proved that this cannot happen in *finite time* in the evolution of closed manifolds.

Theorem (Perelman 2002 [P1]**)** *Let $(M^n, g(t))$ be a closed solution of the Ricci flow for $0 < t < T < \infty$ with initial metric $g(0)$. Then there exists a constant $\kappa > 0$ which only depends on n, T and the initial metric $g(0)$ such that*

$$\frac{Vol_t(B^t_r(p))}{r^n} \geq \kappa$$

for all $t < T$ and $r \leq \sqrt{T}$ in balls $B_r^t(p)$ (with respect to $g(t)$) in which the curvature $|\mathcal{R}(t)|$ is controlled by $1/r^2$. In particular, any rescaling limit $(M, g'(s))$ of $(M, g(t))$ satisfies

$$\frac{Vol_s(B_r^s(p))}{r^n} \geq \kappa$$

for all $0 < s < \infty$ with the same κ as above but this time for all $r < \infty$ in balls $B_r^s(p)$ where $|\mathcal{R}'(s)| \leq 1$. Here $\mathcal{R}'(s)$ denotes the curvature operator with respect to $g'(s)$.

This rules out $\Sigma \times \mathbb{R}$ as a rescaling limit. To explain the analysis behind this in detail would go beyond the scope of this exposition. However, let us mention that the proof of the above lower volume ratio bound is a consequence of the preservation of a functional inequality (like a logarithmic Sobolev inequality) on $(M, g(t))$ during the evolution which in turn is related to preserving ordinary Sobolev inequalities, isoperimetric inequalities and the Poincaré inequality for finite time as long as the curvatures are under control.

As we saw in the first section, this kind of information also controls the diffusion rate of solutions of the heat equation. Therefore one can consider this another more involved situation where analysis (heat diffusion) and geometry interact.

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