Problem 1. Find a solution $u: \mathbb{R} \times (0, \infty) \to \mathbb{R}$ of the heat equation with initial condition $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$. You are allowed to guess (it may be easier than you expect!). Use this solution and the representation formula for solutions derived in lectures in order to determine the integrals
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} \, dz \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-z^2/2} \, dz.$$ Please compute the integrals this way and not by any other method you may have learned in the past.

2 points

Problem 2. Let $b \in \mathbb{R}$, $k > 0$ and $a \in \mathbb{R}^n$ be given. Derive a representation formula for solutions of the Cauchy problem with initial data $g: \mathbb{R}^n \to \mathbb{R}$ of the equation
$$\frac{\partial u}{\partial t} - k \Delta u = a \cdot Du + bu.$$ (Hint: By using suitable transformations of variables one can construct from $u$ a solution $v$ of the standard heat equation to which one can then apply the standard representation formula from lectures. Do not forget to transform back from $v$ to $u$ and also to check your result!)

4 points

Problem 3. Let $u$ be a solution of the problem
$$u_t = u_{xx}$$ for $0 < x < 1$, $0 < t < \infty$, which satisfies the boundary condition $u(0, t) = u(1, t) = 0$ for $0 < t < \infty$ and initial condition $u(x, 0) = 4x(1-x)$ for $0 \leq x \leq 1$.

Show:
(a) $0 < u(x,t) < 1$ for all $0 < x < 1$ and all $t > 0$.
(b) $u(x,t) = u(1-x,t)$ for all $0 \leq x \leq 1$ and all $t \geq 0$.

(Hint: For (a) use both the weak and the strong maximum principle, and for (b) use the uniqueness of solutions proved in lectures.)

4 Punkte

To be returned on Tuesday 04.06.2019 at 2 p.m. (sharp) (14 Uhr s.t.) to Klaus Ecker’s tutorial box