Problem 1. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be continuous and bounded. In lectures it was proved that the function \( u \) defined by

\[
    u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(y)e^{-\frac{|x-y|^2}{4t}} \, dy
\]

for all \( x \in \mathbb{R}^n \) and \( t > 0 \) satisfies the heat equation and attains the initial data \( g \) continuously.

Assume now that in addition we have \( g \geq 0 \) on all of \( \mathbb{R}^n \) and \( g(y_0) > 0 \) for at least one \( y_0 \in \mathbb{R}^n \). Show directly from the above representation formula that \( u(x,t) > 0 \) for all \( x \in \mathbb{R}^n \) and all \( t > 0 \). Do not use any maximum principle arguments.

2 points

Problem 2. Let \( U \subset \mathbb{R}^n \) be open and bounded, \( T \in (0,\infty) \) and let \( u \) be a solution in \( U \times (0,T) \) of the equation

\[
    \left( \frac{\partial}{\partial t} - \Delta \right) u(x,t) = a(x,t) \cdot Du(x,t) + b(x,t)u(x,t),
\]

where \( a : \bar{U} \times [0,T] \to \mathbb{R}^n \) and \( b : \bar{U} \times [0,T] \to \mathbb{R} \) are continuous. Assume furthermore that \( u(x,t) = 0 \) for all \( (x,t) \in \partial U \times [0,T] \).

Prove: If \( u(x,0) \leq 0 \) for all \( x \in \bar{U} \) then \( u(x,t) \leq 0 \) holds for all \( (x,t) \in \bar{U} \times [0,T] \).

(Hint: Set \( u_\epsilon(x,t) = u(x,t) - \epsilon(t + \delta) \) where \( 0 < \delta \leq (2\max_{\bar{U} \times [0,T]} |b| + 1)^{-1} \). Show that \( u_\epsilon(x,0) \leq -\epsilon \delta < 0 \) for all \( x \in \bar{U} \). Use the differential equation for \( u_\epsilon \) to prove by contradiction that there cannot be a point \( (x_0, t_0) \in U \times (0,\delta) \) where

\[
    0 = u_\epsilon(x_0, t_0) = \max_{U \times [0, t_0]} u_\epsilon
\]

holds. Infer from this that \( u_\epsilon(x,t) < 0 \) for all \( (x,t) \in \bar{U} \times [0,\delta] \) and hence \( u(x,t) \leq 0 \) for all \( (x,t) \in \bar{U} \times [0,\delta] \). Conclude now that \( u(x,t) \leq 0 \) for all \( (x,t) \in \bar{U} \times [0,T] \).)

4 points

Please turn over!
**Problem 3.** Let $U \subset \mathbb{R}^n$ be open and bounded with $\partial U \in C^1$. Define the *mean value* of an integrable function $f : U \to \mathbb{R}$ by

$$\bar{f} = \frac{1}{\text{vol}(U)} \int_U f(x) \, dx.$$ 

Let $u : U \times (0, \infty) \to \mathbb{R}$ be a solution of the heat equation with initial condition $u(x, 0) = u_0(x)$ for all $x \in \bar{U}$ and given function $u_0 \in C^0(\bar{U})$ and with (Neumann-) boundary condition $\frac{\partial u}{\partial \nu}(x, t) = 0$ for all $(x, t) \in \partial U \times (0, \infty)$. As usual, since we are dealing with classical solutions in this lecture subject, you may assume that $u(\cdot, t) \in C^2(\bar{U})$ for all $t > 0$, that $\frac{\partial u}{\partial t}(\cdot, t)$ is continuous on $\bar{U}$ for all $t > 0$ and that $u(\cdot, t)$ attains its initial data $u_0$ continuously on $\bar{U}$.

(a) Prove: The mean value $\bar{u}(t)$ of $u(\cdot, t)$ satisfies $\frac{d}{dt} \bar{u}(t) = 0$ for all $t > 0$. (Hint: Use the divergence theorem. Please provide all details of your calculations. In particular justify why you are allowed to differentiate with respect to $t$ under the integral sign in this case. Do not just refer to lecture material on conservation laws.)

Open and bounded sets $U \subset \mathbb{R}^n$ with $\partial U \in C^1$ satisfy a Poincaré inequality which will be proved in the lecture subject on Partial Differential Equations 2. (see Evans, *Partial Differential Equations*, section 5.8.1, Theorem 1 with $p = 2$). This states that there is a constant $\delta > 0$ (which depends only on $n$ and $U$) such that for all functions $f \in C^1(\bar{U})$ there holds

$$\delta \int_U (f(x) - \bar{f})^2 \, dx \leq \int_U |Df(x)|^2 \, dx.$$ 

(b) By differentiating a suitable integral expression with respect to $t$ and using part (a) as well as integration by parts and the Poincaré inequality prove the following: There are constants $c_1, c_2 > 0$ (these are only allowed to depend on $n$ and $U$) such that for all $t \geq 0$ the inequality

$$\int_U (u(x, t) - \bar{u}(0))^2 \, dx \leq c_1 e^{-c_2 t}$$

holds. (This says in particular that $u(\cdot, t)$ approaches the mean value of its initial data in an integral sense as $t \to \infty$. This is called *heat diffusion*.)

4 points

To be returned on Tuesday 11.06.2019 at 2 p.m. (sharp) (14 Uhr s.t.) to Klaus Ecker's tutorial box.