Fachbereich Mathematik und Informatik
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Differentialgeometrie II SS 2019

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Tutorial Sheet 2

In all of the following, M denotes an n-dimensional smooth manifold.

Exercise 1. Let $T : M \to T^1_1 M$ be a smooth tensor field and $X : M \to TM$ a smooth vector field. Prove that

$$\pounds_X \mathrm{tr}\, T = \mathrm{tr}\, \pounds_X T.$$

3 marks

Exercise 2. Let S be an m-dimensional smooth manifold and $F : S \to M$ an embedding. Show the following:

(a) If $m = n$, then $F : S \to F(S)$ is a diffeomorphism. Therefore, $F(S)$ is an open submanifold.

(b) If S is compact, then $F(S)$ is an embedded submanifold, and F is *proper* in the following sense: Whenever $K \subset M$ is compact, $F^{-1}(K)$ is compact.

3 marks

Exercise 3. Let $X, Y : M \to TM$ be smooth vector fields such that $[X, Y] \equiv 0$ and suppose that there exists a $p_0 \in M$ such that $X(p_0)$ and $Y(p_0)$ are linearly independent.

(a) Verify that there exists a coordinate system $\varphi|_U$ in a neighbourhood U of p_0 such that $\varphi(p_0) = 0$ and $\{X(p_0), Y(p_0), \partial_3|_{p_0}, \dots, \partial_n|_{p_0}\}$ are linearly independent.

(b) Let φ be as in part (a) and write $\{X_t\}$ and $\{Y_t\}$ for the local flows of X and Y. Define the mapping

$$\begin{aligned} F : \,]{-\delta}, \delta[^n &\to M \\ (t^1, \dots, t^n) &\mapsto X_{t^1}(Y_{t^2}(\varphi^{-1}(0, 0, t^3, \dots, t^n))), \end{aligned}$$

where $\delta > 0$ is small enough so that this function is well-defined and thus smooth. Show that

- $F(0) = p_0$;
- $\partial_1 F = X \circ F$ and $\partial_2 F = Y \circ F$; and
- for $j > 2$, $\partial_j F(0) = \partial_j\big|_{p_0}$.

(c) Deduce that there exists a $\delta_0 \leq \delta$ such that $F|_{]-\delta_0, \delta_0[^n}$ is a diffeomorphism onto its image. Hence, $\varphi'|_{U'} := F|^{-1}_{]-\delta_0, \delta_0[^n}$ is a coordinate system with respect to which $X|_{U'} = \partial'_1$ and $Y|_{U'} = \partial'_2$.

4 marks

To be handed in by 10 a.m. on 10/05/2019.