Fachbereich Mathematik und Informatik
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Differentialgeometrie II SS 2019

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Tutorial Sheet 3

Exercise 1 (special linear group). Let $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det(A) = 1\}$.

(a) Show that $\mathrm{SL}(n,\mathbb{R})$ is an embedded submanifold of $\mathrm{GL}(n,\mathbb{R})$, and

$$T_I\mathrm{SL}(n,\mathbb{R}) \simeq \{A \in M_{n\times n}(\mathbb{R}) : \operatorname{tr} A = 0\}.$$

(b) Verify that $\mathrm{SL}(n,\mathbb{R})$ is closed under matrix multiplication, and the multiplication map

$$\mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$$

and inversion map $\mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ are smooth.

4 marks

Exercise 2. An n-dimensional manifold M is said to be *parallelisable* if there exist n smooth vector fields $\{X_1, \ldots, X_n\}$ such that for all $x \in M$,

$$T_xM = \operatorname{span}\{X_1(x), \ldots, X_n(x)\}.$$

Show that if G is a Lie group, then it is parallelisable.
Hint: For $g \in G$, consider the differential of the left multiplication map

$$\begin{aligned} \lambda_g : G &\to G \\ h &\mapsto g \cdot h \end{aligned}$$

at the identity element $e \in G$, i.e. $\mathrm{d}_e\lambda_g$, and a fixed basis $\{b_i\}$ of T_eG.

3 marks

Exercise 3 (torsion tensor). Let M be an n-dimensional manifold equipped with an affine connection L. In every coordinate system $\varphi|_U$, we may construct the following quantity from the coefficients of L:

$$\Omega^i{}_{jk} := \frac{1}{2}\left(L^i_{jk} - L^i_{kj}\right)$$

(a) Show that the tensor field $\Omega := \sum_{i,j,k} \Omega^i{}_{jk}\partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$ is well-defined, i.e. this definition is independent of the choice of coordinate system φ.

(b) Verify that for vector fields X and Y, we have the following equality:

$$Y \llcorner (X \llcorner \Omega) = \frac{1}{2}(\nabla_Y X - \nabla_X Y - [Y,X]).$$

3 marks

To be handed in by 10 a.m. on 17/05/2019.