Throughout this tutorial sheet, we use the Einstein summation convention.

**Exercise 1** (scaling & translation). Let \( \alpha = \alpha_{i_1 \ldots i_n} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_n} \in \Gamma^0_r \mathbb{R}^n \).

(a) For \( r > 0 \), let \( \sigma_r : \mathbb{R}^n \to \mathbb{R}^n \) be such that \( \sigma_r(x) = r \cdot x \). Show that for \( x \in \mathbb{R}^n \),

\[
\sigma^*_r(\alpha)(x) = r^i \cdot \alpha_{i_1 \ldots i_n}(rx) \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_n}|_x .
\]

(b) For \( y \in \mathbb{R}^n \), let \( \rho_y : \mathbb{R}^n \to \mathbb{R}^n \) be such that \( x \mapsto x + y \). Show that for \( x \in \mathbb{R}^n \),

\[
\rho^*_y(\alpha)(x) = \alpha_{i_1 \ldots i_n}(x + y) \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_n}|_x .
\]

**3 marks**

**Exercise 2.** Let \( T^n \) denote the n-torus, \( \pi_n : \mathbb{R}^n \to T^n \) the canonical projection (cf. Example 8.9) and \( \rho_y (y \in \mathbb{R}^n) \) the translation mapping from Exercise 1.

(a) Verify that \( d_x \pi_n : T^n_1 \to T^n_{\pi_n(x)}T^n \) is an isomorphism for all \( x \in \mathbb{R}^n \), and \( \pi_n(x) = \pi_n(y) \) iff \( x - y \in \mathbb{Z}^n \). In particular, for all \( q \in \mathbb{Z}^n \), \( \pi_n \circ \rho_q = \pi_n \).

(b) Let \( \alpha \in \Gamma^0_1 T^n \) and define \( \alpha := \pi_n^* \alpha \). Show that for all \( q \in \mathbb{Z}^n \), \( \rho^*_q \alpha = \alpha \).

(c) Let \( \beta \in \Gamma^0_1 \mathbb{R}^n \) be such that \( \rho^*_q \beta = \beta \) for all \( q \in \mathbb{Z}^n \). Show that the assignment \( \overline{\beta}(p) := (\delta_x \pi_n)^{-1} \beta(x) \) for \( p \in T^n \) and \( x \in \pi_n^{-1}(p) \) is well-defined and yields a smooth tensor field \( \overline{\beta} \in \Gamma^0_1 T^n \) such that \( \pi_n^* \overline{\beta} = \beta \).

**4 marks**

**Exercise 3.** Let \( G \) be a Lie group with identity element \( e \). Recall that \( X \in \Gamma^0_1 G \) is said to be left-invariant if for all \( g, h \in G \), \( X(g \cdot h) = d_h \lambda_g(X(h)) \), where \( \lambda_g : G \to G \) is the left multiplication map \( h \mapsto g \cdot h \).

(a) Show that \( X \in \Gamma^0_1 G \) is left-invariant iff \( X(g) = d_e \lambda_g(v) \) for some \( v \in T_e G \).

(b) Let \( \gamma_v : \mathbb{R} \to G \) be the unique maximal integral curve of the vector field \( g \mapsto d_e \lambda_g(v) \) with \( v \in T_e G \) and \( \gamma_v(0) = e \). Verify that \( \gamma_v(t) = \gamma_{tv}(1) \) for all \( t \in \mathbb{R} \).

**Hint:** Consider the two curves \( s \mapsto \gamma_v(st) \) and \( s \mapsto \gamma_{tv}(s) \) for fixed \( t \).

**3 marks**

**Remark.** The mapping exp : \( T_e G \to G \) such that \( v \mapsto \gamma_v(1) \) is called the exponential map.

To be handed in by 10 a.m. on 24/05/2019.