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Differentialgeometrie II SS 2019

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Tutorial Sheet 4

Throughout this tutorial sheet, we use the Einstein summation convention.

Exercise 1 (scaling & translation)**.** Let $\alpha = \alpha_{i_1 \dots i_s} \cdot \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_s} \in \Gamma^0_s \mathbb{R}^n$.

(a) For $r > 0$, let $\sigma_r : \mathbb{R}^n \to \mathbb{R}^n$ be such that $\sigma_r(x) = r \cdot x$. Show that for $x \in \mathbb{R}^n$,

$$(\sigma_r^* \alpha)(x) = r^s \cdot \alpha_{i_1 \dots i_s}(rx) \cdot \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_s}\big|_x .$$

(b) For $y \in \mathbb{R}^n$, let $\rho_y : \mathbb{R}^n \to \mathbb{R}^n$ be such that $x \mapsto x + y$. Show that for $x \in \mathbb{R}^n$,

$$(\rho_y^* \alpha)(x) = \alpha_{i_1 \dots i_s}(x + y) \cdot \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_s}\big|_x .$$

3 marks

Exercise 2. Let T^n denote the n-torus, $\pi_n : \mathbb{R}^n \to T^n$ the canonical projection (cf. Example 8.9) and ρ_y ($y \in \mathbb{R}^n$) the translation mapping from Exercise 1.

(a) Verify that $\mathrm{d}_x \pi_n : T_x \mathbb{R}^n \to T_{\pi_n(x)} T^n$ is an isomorphism for all $x \in \mathbb{R}^n$, and $\pi_n(x) = \pi_n(y)$ iff $x - y \in \mathbb{Z}^n$. In particular, for all $q \in \mathbb{Z}^n$, $\pi_n \circ \rho_q = \pi_n$.

(b) Let $\alpha \in \Gamma^0_s T^n$ and define $\underline{\alpha} := \pi_n^* \alpha$. Show that for all $q \in \mathbb{Z}^n$, $\rho_q^* \underline{\alpha} = \underline{\alpha}$.

(c) Let $\beta \in \Gamma^0_s \mathbb{R}^n$ be such that $\rho_q^* \beta = \beta$ for all $q \in \mathbb{Z}^n$. Show that the assignment $\overline{\beta}(p) := \underline{(\delta_x \pi_n)^{-1} \beta(x)}$ for $p \in T^n$ and $x \in \pi_n^{-1}(p)$ is well-defined and yields a smooth tensor field $\overline{\beta} \in \Gamma^0_s T^n$ such that $\pi_n^* \overline{\beta} = \beta$.

4 marks

Exercise 3. Let G be a Lie group with identity element e. Recall that $X \in \Gamma^1_0 G$ is said to be *left-invariant* if for all $g, h \in G$, $X(g \cdot h) = \mathrm{d}_h \lambda_g(X(h))$, where $\lambda_g : G \to G$ is the left multiplication map $h \mapsto g \cdot h$.

(a) Show that $X \in \Gamma^1_0 G$ is left-invariant iff $X(g) = \mathrm{d}_e \lambda_g(v)$ for some $v \in T_e G$.

(b) Let $\gamma_v : \mathbb{R} \to G$ be the unique maximal integral curve of the vector field $g \mapsto \mathrm{d}_e \lambda_g(v)$ with $v \in T_e G$ and $\gamma_v(0) = e$. Verify that $\gamma_v(t) = \gamma_{tv}(1)$ for all $t \in \mathbb{R}$.
Hint: Consider the two curves $s \mapsto \gamma_v(st)$ and $s \mapsto \gamma_{tv}(s)$ for fixed t.

3 marks

Remark. The mapping $\exp : T_e G \to G$ such that $v \mapsto \gamma_v(1)$ is called the *exponential map*.

To be handed in by 10 a.m. on 24/05/2019.