Fachbereich Mathematik und Informatik
Freie Universität Berlin

Differentialgeometrie II SS 2019

Ahmad Afuni

Tutorial Sheet 5

Throughout this tutorial sheet, M is an affinely connected manifold and for every smooth curve $\gamma : I \longrightarrow M$ on an interval $I \ni t_0$, write $(t, v_0) \mapsto P^t_{\gamma,t_0} v_0$ for the associated parallelism. We also use the Einstein summation convention.

Exercise 1. Let $\gamma : I \longrightarrow M$ be a smooth curve on an interval $I \ni t_0$.

(a) Show that if $S \in (T^{r_1}_{s_1}M)_{\gamma(t_0)}$ and $T \in (T^{r_2}_{s_2}M)_{\gamma(t_0)}$, then for all $t \in I$,

$$P^t_{\gamma,t_0}(S \otimes T) = P^t_{\gamma,t_0}S \otimes P^t_{\gamma,t_0}T.$$

(b) Show that if $T \in (T^r_sM)_{\gamma(t_0)}$ and $S \in (T^s_rM)_{\gamma(t_0)}$, then for all $t \in I$,

$$(P^t_{\gamma,t_0}T, P^t_{\gamma,t_0}S) = (T, S).$$

Hint: Since parallel displacement is linear, it suffices to consider basis elements in place of S and T.

3 marks

Exercise 2. Let γ be as in Exercise 1 and suppose $\{b_i\}^n_{i=1}$ is a basis for $T_{\gamma(t_0)}M$ with dual basis $\{b^i\}^n_{i=1}$. Using parallelism, we obtain for each $i \in \{1, \ldots, n\}$ a parallel vector field $\beta_i \in {}_\gamma\Gamma^1_0M$ along γ such that $\beta_i(t) := P^t_{\gamma,t_0}b_i$, as well as a smooth covector field $\beta^i \in {}_\gamma\Gamma^0_1M$ such that $\beta^i(t) = {}^t(P^t_{\gamma,t_0})^{-1}b^i$.

(a) Verify that $\{\beta_i(t)\}^n_{i=1}$ and $\{\beta^i(t)\}^n_{i=1}$ are dual bases for $T_{\gamma(t)}M$ and $T^*_{\gamma(t)}M$.

(b) Let $X \in {}_\gamma\Gamma^1_0M$. For each $t \in I$, we may write $X(t) = X^i(t)\beta_i(t)$ for suitable $X^i \in C^\infty(I)$. Show that for all $t \in I$,

$$\frac{\delta X}{\delta t}(t) = \dot{X}^i(t)\beta_i(t).$$

(c) Verify that

$$\frac{\delta X}{\delta t}(t_0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (P^t_{\gamma,t_0})^{-1}(X(t)).$$

Hint: Use the preceding part and the fact that $X^i(t) = (\beta^i(t), X(t))$.

4 marks

Exercise 3. Let $\underset{1}{L}$ and $\underset{2}{L}$ be two affine connections on M. In every coordinate system $\varphi|_U$, we may construct the following quantity from the coefficients of $\underset{1}{L}$ and $\underset{2}{L}$:

$$A^i{}_{jk} := \underset{1}{L}{}^i_{jk} - \underset{2}{L}{}^i_{jk}.$$

Verify that the tensor field $A := A^i{}_{jk}\partial_i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$ is well-defined, i.e. independent of the choice of coordinate system φ.

3 marks

To be handed in by 10 a.m. on 31/05/2019.