Fachbereich Mathematik und Informatik
Freie Universität Berlin

Differentialgeometrie II SS 2019

Ahmad Afuni

Tutorial Sheet 6

Throughout this tutorial sheet, λ_g denotes left translation by g in a Lie group G.

Exercise 1. Let G be an n-dimensional Lie group and $\{\beta_i\}_{i=1}^n$ a basis for the set of all left-invariant vector fields.[1] Suppose G is equipped with the unique affine connection such that $\nabla\beta_i \equiv 0$ for all i.

(a) Verify that if $\{\gamma_i\}_{i=1}^n$ is any other basis for the set of left-invariant vector fields, then $\nabla\gamma_i \equiv 0$.

(b) Show that for any smooth curve $\gamma : I \ni t_0 \longrightarrow G$, we have that

$$P^t_{\gamma,t_0} = \mathrm{d}_e\lambda_{\gamma(t)} \circ \mathrm{d}_{\gamma(t_0)}\lambda_{\gamma(t_0)^{-1}}.$$

Hint: You might find it helpful to act the right-hand side on $\beta_i(\gamma(t_0))$.

3 marks

Exercise 2. Let $T^1 = \mathbb{R}/\mathbb{Z}$ denote the 1-torus, here considered a Lie group, with canonical projection homomorphism $\pi : \mathbb{R} \longrightarrow T^1$. Define $\frac{\mathrm{d}}{\mathrm{d}\theta} \in \Gamma^1_0 T^1$ such that $\frac{\mathrm{d}}{\mathrm{d}\theta}\big|_{\pi(x)} = \mathrm{d}_{\pi(0)}\lambda_{\pi(x)}(\mathrm{d}_0\pi(\frac{\partial}{\partial x^1}\big|_0))$.

(a) Show that for all $x \in \mathbb{R}$, $\mathrm{d}_x\pi(\frac{\partial}{\partial x^1}\big|_x) = \frac{\mathrm{d}}{\mathrm{d}\theta}\big|_{\pi(x)}$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}\theta}$ coïncides with the coordinate basis fields of the charts of T^1.

(b) Suppose T^1 is equipped with the unique affine connection such that

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}\theta}}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}.$$

Let $\gamma : I \longrightarrow T^1$ be a smooth curve on an interval I with $0 \in I$ and $\mathring{I} \neq \emptyset$ and $\overline{\gamma} : I \longrightarrow \mathbb{R}$ any *lift* of γ, i.e. a smooth curve such that $\gamma = \pi \circ \overline{\gamma}$. Show that a vector field $X \in {}_\gamma\Gamma^1_0 T^1$ of the form $X(t) = f(t) \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\big|_{\gamma(t)}$ is parallel along γ if and only if f satisfies the ODE

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\mathrm{d}\overline{\gamma}}{\mathrm{d}t} \cdot f = 0.$$

(c) Deduce that parallel displacement from $\gamma(0)$ along γ is given by

$$P^t_{\gamma,0} = e^{\bar{\gamma}(0)-\bar{\gamma}(t)} \cdot \mathrm{d}_{\pi(0)}\lambda_{\gamma(t)} \circ \mathrm{d}_{\gamma(0)}\lambda_{\gamma(0)^{-1}}$$

4 marks

[1] From the last sheet, we know that the mapping $T_eG \longrightarrow \{\text{left-invariant vector fields on } G\}$ such that $v \mapsto (g \mapsto \mathrm{d}_e\lambda_g(v))$ is an isomorphism.

Exercise 3. Let M and N be smooth manifolds and $f : M \longrightarrow N$ a smooth mapping. Two vector fields $X \in \Gamma^1_0 M$ and $\overline{X} \in \Gamma^1_0 N$ are said to be *f-related*, written $X \underset{f}{\sim} \overline{X}$, if for all $p \in M$,

$$\overline{X}(f(p)) = \mathrm{d}_p f(X(p)).$$

(a) Show that if $X, Y \in \Gamma^1_0 M$ and $\overline{X}, \overline{Y} \in \Gamma^1_0 N$ are such that $X \underset{f}{\sim} \overline{X}$ and $Y \underset{f}{\sim} \overline{Y}$, then

$$[X, Y] \underset{f}{\sim} [\overline{X}, \overline{Y}].$$

Hint: You may approach this either locally by writing out what it means for vector fields to be f-related and using the local expression for $[X, Y]$, or by showing that

$$\partial_{\mathrm{d}_p f([X,Y])} \psi = \partial_{[\overline{X},\overline{Y}](f(p))} \psi$$

for all $\psi \in C^\infty(N)$. In the latter case, you may use the identity

$$\partial_{\mathrm{d}_p f(X(p))} \psi = \partial_{X(p)}(\psi \circ f).$$

(b) Deduce that if G is a Lie group and $X, Y \in \Gamma^1_0 G$ are left-invariant, so is $[X, Y]$.

3 marks

To be handed in by 10 a.m. on 07/06/2019.