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Differentialgeometrie II SS 2019

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Tutorial Sheet 7

Exercise 1. Let G be an n-dimensional Lie group equipped with the canonical left-invariant affine connection as in Exercise 1 of Tutorial Sheet 6.

(a) Verify that the unique path $c : \mathbb{R} \to G$ with $c(0) = g$ and $\dot{c}(0) = \mathrm{d}_e\lambda_g(v)$, $v \in T_eG$, is given by

$$c(t) = \lambda_g \exp(tv),$$

where $\exp : T_eG \to G$ is the exponential map associated with the Lie group (see Tutorial Sheet 4).

(b) Suppose that $\phi : G \to H$ is a smooth homomorphism of Lie groups G and H with identities e_G and e_H respectively and write $\exp_G$ and $\exp_H$ for the respective exponential maps. Show that the following diagram commutes:

$$\begin{array}{ccc} T_{e_G}G & \xrightarrow{\exp_G} & G \\ \mathrm{d}_{e_G}\phi \downarrow & & \downarrow \phi \\ T_{e_H}H & \xrightarrow[\exp_H]{} & H \end{array}$$

(c) Let $b_i := \mathrm{d}_0\pi_n(\frac{\partial}{\partial x^i}\big|_0) \in T_{\pi_n(0)}T^n$, where $\pi_n : (\mathbb{R}^n, +) \to (T^n, +)$ is the canonical projection homomorphism and $(x^1, \ldots, x^n)$ are the canonical coordinates on $\mathbb{R}^n$. Deduce that the unique path $c : \mathbb{R} \to T^n$ with $c(0) = \pi_n(p)$ and $\dot{c}(0) = v^i b_i$ is given by

$$c(t) = \pi_n(p + tv^i e_i).$$

Hints: For the first two parts, you may want to consider the respective uniqueness theorems as well as the fact that $t \mapsto \exp_G(tv)$ is the unique integral curve $\gamma : \mathbb{R} \to G$ of the left-invariant vector field

$$G \ni g \mapsto X_v(g) := \mathrm{d}_e\lambda_g(v)$$

satisfying $\gamma(0) = e_G$.

3+3 marks

Exercise 2. Let M be a Riemannian manifold and set

$$\Omega_{p,q} := \{\gamma : [0,1] \to M \text{ piecewise smooth} : \gamma(0) = p, \gamma(1) = q\}.$$

Recall that the *arclength* functional $L : \Omega_{p,q} \to \mathbb{R}$ is given by

$$L(\gamma) = \int_0^1 |\dot{\gamma}|.$$

The *energy* functional $E : \Omega_{p,q} \to \mathbb{R}$ is defined by

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}|^2.$$

(a) Show that the inequality $L(\gamma) \leq \sqrt{2E(\gamma)}$ holds for all $\gamma \in \Omega_{p,q}$ with equality holding iff $|\dot{\gamma}|$ is constant on each subinterval where it is smooth.

(b) Suppose that $\gamma^* \in \Omega_{p,q}$ minimises L, i.e. $L(\gamma^*) \leq L(\gamma)$ for all $\gamma \in \Omega_{p,q}$, and is such that $|\dot{\gamma}^*|$ is constant on each subinterval where it is smooth. Show that γ^* also minimises E.

3 marks

Exercise 3. Let M be a Riemannian manifold. The *unit sphere bundle* is defined to be

$$SM := \{v \in TM : |v|^2 = 1\}.$$

(a) Show that SM is an embedded submanifold of TM.
Hint: Level set theorem!

(b) Show that if M is compact, then so is SM.

(c) Suppose that M is compact and equipped with the Levi-Cività connection. Show that if $\gamma_v : I_v \to M$ is the unique maximal path with $\dot{\gamma}_v(0) = v \in T_pM$, then $I_v = \mathbb{R}$.

4+3 marks

To be handed in by 10 a.m. on 14/06/2019.