Fachbereich Mathematik und Informatik
Freie Universität Berlin

Differentialgeometrie II SS 2019

Ahmad Afuni

Tutorial Sheet 8

In all of the following, (M, g) is an n-dimensional Riemannian manifold.

Exercise 1. Let $p \in M$, $\{v_i\}_{i=1}^n$ an orthonormal basis for T_pM and $\sigma_{ij} := \operatorname{span}\{v_i, v_j\}$.

(a) Show that the Ricci tensor and scalar curvature satisfy the following relations:

$$(\mathrm{Ric}, v_i \otimes v_i) = \sum_{j \neq i} K(\sigma_{ij}); \qquad \mathrm{scal} = \sum_{i=1}^{n} \sum_{j \neq i} K(\sigma_{ij}).$$

(b) Let $\lambda > 0$ and define the *rescaled metric* $g^\lambda := \lambda g$. Show that the curvature quantities K^λ, Ric^λ and scal^λ of g^λ are related to those of g by

$$K^\lambda = \lambda^{-1} K; \qquad \mathrm{Ric}^\lambda = \mathrm{Ric}; \qquad \mathrm{scal}^\lambda = \lambda^{-1}\mathrm{scal}.$$

3 marks

Exercise 2. The Riemannian manifold (M, g) is said to be an *Einstein manifold* if there is a function $f \in C^\infty(M)$ such that $\mathrm{Ric} = f \cdot g$.

(a) Show that if (M, g) is an Einstein manifold, then $\mathrm{Ric} = \frac{\mathrm{scal}}{n} g$.

(b) Suppose (M, g) is a connected Einstein manifold of dimension $n > 2$. Verify that scal is constant (*Hint*: Consider the Einstein tensor!).

4 marks

Exercise 3. Suppose (N, g_N) is a Riemannian manifold. A diffeomorphism

$$f : (M, g) \to (N, g_N)$$

is said to be an *isometry* if $f^* g_N = g$, i.e. if for all $p \in M$ and $v, w \in T_pM$,

$$\langle v, w \rangle_g = \left\langle \mathrm{d}_p f(v), \mathrm{d}_p f(w) \right\rangle_{g_N}.$$

(a) Show that f is a *metric space isometry*, i.e. if d_M and d_N denote the metrics induced on M and N respectively, we have that for all $p, q \in M$,

$$d_N(f(p), f(q)) = d_M(p, q).$$

(b) A vector field $X \in \Gamma_0^1 M$ is said to be a *Killing field* if it satisfies *Killing's equation*:

$$\pounds_X g = 0.$$

This is equivalent to the condition $X_t^* g = g$ for every t, where $X_t : Q(t) \to Q(-t)$ is the local flow generated by X. Show that Killing's equation is also equivalent to the condition that $\nabla X^\flat$ be skew-symmetric, i.e.

$$X_{i|j} + X_{j|i} \equiv 0.$$

(c) Show that $X = x^i \partial_i$ is a Killing field on $(\mathbb{H}^n, g_{\mathrm{hyp}})$.

3+4 marks

To be handed in by 10 a.m. on 21/06/2019.