Exercise 1. Suppose \((M, g)\) is a Riemannian manifold and \(\varphi|_{U(p)} = (x^1, \ldots, x^n)\) are geodesic normal coordinates about \(p \in M\), where \(U(p) := \exp_p(B(0_p, \text{inj}_p))\).

(a) Show using the Gauß Lemma that in these coordinates, \(g_{ij}x^j = x^i\).

(b) Recall that the geodesic distance from \(p \in M\) is given by the mapping
\[
\rho(p, \cdot) : U(p) \to \mathbb{R}; \quad \rho(p, q) = |\varphi(q)|.
\]
Verify that in geodesic normal coordinates about \(p\),
\[
d\rho(p, \cdot) = \sum_{i=1}^n \frac{x^i}{\rho(p, \cdot)} \, dx^i, \quad \nabla \rho(p, \cdot) = \sum_{i=1}^n \frac{x^i}{\rho(p, \cdot)} \partial_i,
\]
and that \(|\nabla \rho(p, \cdot)| \equiv |d\rho(p, \cdot)| \equiv 1.\]

3 marks

Exercise 2. Suppose \((M, g)\) is a Riemannian manifold and \(p, q \in M\).

(a) Suppose \(c \in \Omega_{p,q}\) is a length-minimising curve, i.e. \(L(c) = d(p, q)\). Show that whenever \(0 < t_1 < t_2 < 1\),
\[
d(c(t_1), c(t_2)) = L(c|_{[t_1, t_2]}).
\]

(b) Suppose \(M\) is complete and noncompact, i.e. for fixed \(p \in M\), there exists a sequence \(\{q_i\}_{i=1}^\infty\) such that \(d(p, q_i) \to \infty\). Show that for each \(p \in M\), there exists a unit speed geodesic \(c : [0, \infty[ \to M\) such that \(c(0) = p\) and for all \(t > 0\),
\[
d(p, c(t)) = t = L(c|_{[0, t]}).
\]

Hint: For a sequence as above, we may write \(q_i = \exp_p(d(p, q_i)v_i)\) for \(\{v_i\} \subset SM \cap T_pM\). Consider \(c(t) = \exp_p(tv)\) with \(v\) a (subsequential) limit of the \(\{v_i\}\) and use (a)!

4 marks

Exercise 3. Let \((M, g)\) be a Riemannian manifold and \(p, q := \exp_p(v) \in M\) points that are not conjugate. Show that for any \((w_1, w_2) \in T_pM \oplus T_qM\), there exists a unique Jacobi field \(J\) along the geodesic \(\gamma_v\) such that \(J(0) = w_1\) and \(J(1) = w_2\).

Hint: Consider the injectivity of the linear mapping \(T_pM \oplus T_pM \to T_pM \oplus T_qM\) such that \((v_1, v_2) \mapsto (J_{v_1, v_2}(0), J_{v_1, v_2}(1))\) and compare dimensions!

3 marks
Exercise 4. Let $N_1 := F_1(S_1)$ and $N_2 := F_2(S_2)$ be submanifolds of $M$ and suppose $c : [0, 1] \to M$ is a geodesic such that $c(0) \in N_1$, $c(1) \in N_2$ and

$$L(c) = d(N_1, N_2) = \inf_{(p,q) \in N_1 \times N_2} d(p, q).$$

Show that for each $v \in T_{c(0)}S_1$ and $w \in T_{c(1)}S_2$,

$$\langle \dot{c}(0), dF_1(v) \rangle = \langle \dot{c}(1), dF_2(w) \rangle = 0.$$

*Hint:* First variation formula with an appropriate variation!

4 marks

To be handed in by 10 p.m. on 02/07/2019.