Fachbereich Mathematik und Informatik
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Differentialgeometrie II SS 2019

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Tutorial Sheet 9

Exercise 1. Suppose (M, g) is a Riemannian manifold and $\varphi|_{U(p)} = (x^1, \ldots, x^n)$ are geodesic normal coordinates about $p \in M$, where $U(p) := \exp_p(B(0_p, \mathrm{inj}_p))$.

(a) Show using the Gauß Lemma that in these coordinates, $g_{ij}x^j = x^i$.

(b) Recall that the *geodesic distance* from $p \in M$ is given by the mapping

$$\rho(p, \cdot) : U(p) \longrightarrow \mathbb{R}; \qquad \rho(p, q) = |\varphi(q)|.$$

Verify that in geodesic normal coordinates about p,

$$\mathrm{d}\rho(p, \cdot) = \sum_{i=1}^{n} \frac{x^i}{\rho(p, \cdot)} \mathrm{d}x^i, \qquad \nabla\rho(p, \cdot) = \sum_{i=1}^{n} \frac{x^i}{\rho(p, \cdot)} \partial_i,$$

and that $|\nabla\rho(p, \cdot)| \equiv |\mathrm{d}\rho(p, \cdot)| \equiv 1$.

3 marks

Exercise 2. Suppose (M, g) is a Riemannian manifold and $p, q \in M$.

(a) Suppose $c \in \Omega_{p,q}$ is a length-minimising curve, i.e. $L(c) = d(p, q)$. Show that whenever $0 < t_1 < t_2 < 1$,

$$d(c(t_1), c(t_2)) = L(c|_{[t_1,t_2]}).$$

(b) Suppose M is complete and *noncompact*, i.e. for fixed $p \in M$, there exists a sequence $\{q_i\}_{i=1}^{\infty}$ such that $d(p, q_i) \xrightarrow{i\to\infty} \infty$. Show that for each $p \in M$, there exists a unit speed geodesic $c : [0, \infty[\longrightarrow M$ such that $c(0) = p$ and for all $t > 0$,

$$d(p, c(t)) = t (= L(c|_{[0,t]})).$$

Hint: For a sequence as above, we may write $q_i = \exp_p(d(p, q_i)v_i)$ for $\{v_i\} \subset SM \cap T_pM$. Consider $c(t) = \exp_p(tv)$ with v a (subsequential) limit of the $\{v_i\}$ and use (a)!

4 marks

Exercise 3. Let (M, g) be a Riemannian manifold and $p, q := \exp_p(v) \in M$ points that are *not* conjugate. Show that for any $(w_1, w_2) \in T_pM \oplus T_qM$, there exists a unique Jacobi field J along the geodesic γ_v such that $J(0) = w_1$ and $J(1) = w_2$.

Hint: Consider the injectivity of the linear mapping $T_pM \oplus T_pM \longrightarrow T_pM \oplus T_qM$ such that $(v_1, v_2) \mapsto (J_{v_1,v_2}(0), J_{v_1,v_2}(1))$ and compare dimensions!

3 marks

Exercise 4. Let $N_1 := F_1(S_1)$ and $N_2 := F_2(S_2)$ be submanifolds of M and suppose $c : [0,1] \longrightarrow M$ is a geodesic such that $c(0) \in N_1$, $c(1) \in N_2$ and

$$L(c) = d(N_1, N_2) = \inf_{(p,q) \in N_1 \times N_2} d(p,q).$$

Show that for each $v \in T_{c(0)}S_1$ and $w \in T_{c(1)}S_2$,

$$\langle \dot{c}(0), \mathrm{d}F_1(v) \rangle = \langle \dot{c}(1), \mathrm{d}F_2(w) \rangle = 0.$$

Hint: First variation formula with an appropriate variation!

4 marks

To be handed in by 10 p.m. on 02/07/2019.