Fachbereich Mathematik und Informatik
Freie Universität Berlin

Differentialgeometrie II SS 2019

Ahmad Afuni

Formula Sheet

Non-Riemannian geometry

Tensor transformation rule (coordinates):

$$\bar{T}^{i_1\dots i_r}{}_{j_1\dots j_s} = T^{k_1\dots k_r}{}_{l_1\dots l_s} \cdot \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial x^{j_s}}.$$

Lie derivative of a tensor field:

$$\pounds_X T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} X_t^* T.$$

Lie bracket of vector fields:

$$[X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i.$$

Transformation rule of coefficients of affine connection L:

$$\bar{L}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^j} \cdot \frac{\partial x^r}{\partial \bar{x}^k} \cdot L^p_{qr} + \frac{\partial \bar{x}^i}{\partial x^r} \cdot \frac{\partial^2 x^r}{\partial \bar{x}^j \partial \bar{x}^k}$$

Covariant derivative associated with affine connection L (coordinates):

$$\nabla X = X^i{}_{|j} \partial_i \otimes \mathrm{d}x^j, \qquad X^i{}_{|j} = \partial_j X^i + X^r L^i_{rj}.$$
$$\nabla_Y X = \nabla X \lrcorner Y = X^i{}_{|j} Y^j \partial_i.$$

Absolute derivative along $c : I \longrightarrow M$ (arbitrary frame for TM):

$$\frac{\delta}{\delta t}\left(X^i \cdot \varepsilon_i \circ c\right) = \dot{X}^i \cdot \varepsilon_i \circ c + X^i \cdot \nabla_{\dot{c}} \varepsilon_i.$$

Parallelism along a curve $c : I \ni t_0 \longrightarrow M$:

$$P^t_{c,t_0} : T_{c(t_0)}M \xrightarrow{\simeq} T_{c(t)}M,$$
$$X := (t \mapsto P^t_{c,t_0} v) \text{ unique VF along } c \text{ s.t. } X(t_0) = v,\ \frac{\delta X}{\delta t} \equiv 0.$$

Curvature tensor of L (coordinates):

$$R^i{}_{jkl} = \frac{\partial L^i_{jl}}{\partial x^k} - \frac{\partial L^i_{jk}}{\partial x^l} + L^r_{jl} L^i_{rk} - L^r_{jk} L^i_{rl}.$$

Curvature endomorphism of L (arbitrary frame):

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R^i{}_{jkl} Z^j X^k Y^l \varepsilon_i.$$

Commutation rule for absolute derivatives along a family $\{\gamma(s,\cdot)\}_s$ of curves:

$$\left(\frac{\delta}{\delta s}\frac{\delta}{\delta t} - \frac{\delta}{\delta t}\frac{\delta}{\delta s}\right)X = R(\partial_s\gamma, \partial_t\gamma)X.$$

Lie groups

General form of left-invariant vector field X on a Lie group G:

$$X_v(g) = \mathrm{d}_e\lambda_g(v), \qquad v \in T_eG.$$

Exponential map:

$$\exp = (T_eG \ni v \mapsto \exp(v) = c_v(1) \in G),$$
$$c_v : \mathbb{R} \to G \text{ unique maximal integral curve of } X_v \text{ s.t. } c_v(0) = e.$$

Riemannian geometry

Representation of Riemannian metric and its dual (arbitrary frame):

$$g = g_{ij}\varepsilon^i \otimes \varepsilon^j, \qquad g^* = g^{ij}\varepsilon_i \otimes \varepsilon_j, \qquad (g^{ij}) = (g_{ij})^{-1}.$$

Lengths and angles of vectors (arbitrary frame):

$$\langle v, w\rangle = (g, v \otimes w) = g_{ij}v^iw^j, \qquad |v| = \sqrt{\langle v, v\rangle}.$$

Length of piecewise smooth curve $c : I \to M$ in Riemannian manifold:

$$L(c) = \int_I |\dot{c}|.$$

Metric (distance) on Riemannian manifold:

$$d(p,q) = \inf_{c\in\Omega_{p,q}} L(c),$$
$$\Omega_{p,q} = \{c : [0,1] \to M \text{ piecewise smooth} : c(0) = p, c(1) = q\}.$$

Levi-Cività connection Γ in coordinates:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Geodesic equation (coordinates):

$$\ddot{c} = 0 \Leftrightarrow \frac{\delta\dot{c}}{\delta t} = 0 \Leftrightarrow \ddot{c}^i + \Gamma^i_{jk}\dot{c}^j\dot{c}^k = 0.$$

Exponential map:

$$\exp_p = (T_pM \ni v \mapsto \exp_p(v) = \gamma_v(1) \in M),$$
$$\gamma_v : I_v \to M \text{ unique maximal geodesic s.t. } \gamma_v(0) = p,\ \dot{\gamma}_v(0) = v.$$

Covariant curvature tensor (arbitrary frame):

$$\mathrm{Rm} = R_{ijkl}\varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k \otimes \varepsilon^l; \qquad R_{ijkl} = g_{ir}R^r{}_{jkl}.$$

Sectional curvature of 2-plane $\sigma = \mathrm{span}\{v, w\} \subset T_pM$ (arbitrary frame):

$$K(\sigma) = \frac{\langle R(v,w)w, v\rangle}{|v|^2|w|^2 - \langle v, w\rangle^2} = \frac{R_{ijkl}v^iw^jv^kw^l}{(g_{ik}g_{jl} - g_{il}g_{jk})v^iw^jv^kw^l}.$$

Ricci curvature (arbitrary frame):

$$\mathrm{Ric} = R_{ij}\varepsilon^i \otimes \varepsilon^j, \qquad R_{ij} = R^l{}_{ilj}.$$

Scalar curvature (arbitrary frame):

$$\mathrm{scal} = R_{ij}g^{ij}.$$