EXERCISES ON MEAN CURVATURE FLOW

KLAUS ECKER

1. Let F be defined in terms of $\tilde{F}$ as in (2.4) of [E]. Prove that F satisfies mean curvature flow by using (2.4), the evolution equation for $\tilde{F}$ and the defining ODE system for the tangential diffeomorphisms $\phi(\cdot, t)$. Argue also that the ODE system for $\phi(\cdot, t)$ has a solution. Which property of $\tilde{F}$ is used here in a crucial way?

2. Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$ and $\lambda > 0$. Define $M_\lambda = \lambda M$, that is every $x \in M_\lambda$ is of the form $x = \lambda y$ for some $y \in M$. Prove the identity

$$H_{M_\lambda}(x) = \frac{1}{\lambda} H_M(y).$$

3. Consider a torus in $\mathbb{R}^3$ obtained by rotating the circle of radius $r > 0$ in the xy - plane with centre $(0, \rho, 0)$, $\rho > 0$ about the x - axis.
a. How does one have to choose r and ρ to guarantee that the mean curvature of the torus with respect to the outer unit normal is positive?
b. Does the torus maintain a circular cross section under mean curvature flow?
c. Show that the evolving torus contracts to a circle which is contained inside the original torus. (There ought to be an elementary proof of this. The statement can certainly be proved using more advanced techniques, see exercise 25 below.)

4. Prove the comparison principle as stated in Proposition 2.4 of [E] by showing that for two initially disjoint compact smooth solutions (M_t) and (M'_t)

$$\frac{\partial}{\partial t}|F(p,t) - F'(q,t)|^2 \geq 0$$

holds for all $t > 0$ and $p, q \in M^n$ such that $|F(p,t) - F'(q,t)| = \mathrm{dist}(M_t, M'_t)$ where $F(\cdot, t)$ and $F'(\cdot, t)$ denote the immersion for M_t and M'_t respectively.

5. For an immersed hypersurface M with immersion map F prove the identity

$$\Delta_M F = -H\nu$$

where Δ_M denotes the Laplace Beltrami operator defined in lectures or standard differential geometry classes.

6. For an immersed hypersurface M prove the identity $H = \mathrm{div}_M \nu$.

7. Show that if $M = \{x \in \mathbb{R}^{n+1}, w(x) = 0\}$ for some smooth function $w : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying $Dw(x) \neq 0$ for all $x \in M$ then

$$H = \pm \,\mathrm{div}\left(\frac{Dw}{|Dw|}\right).$$

Date: December 18, 2014.

8. For an immersed hypersurface M prove the inequality

$$|A|^2 \geq \frac{1}{n} H^2$$

where $|A|^2$ denotes the squared norm of the second fundamental form, that is $|A|^2 = \kappa_1^2 + \ldots \kappa_n^2$. (Hint: Use the Cauchy-Schwarz inequality.)

9. Prove (2.10) in [E] from the relevant identities preceding it.

10. Suppose u given by $u(x,t)$ solves the graph equation for mean curvature flow for $x \in \mathbb{R}^n$ and $t > 0$. For $\lambda > 0$ define u^λ by

$$u^\lambda(y,s) = \frac{1}{\lambda} u(\lambda y, \lambda^2 s).$$

Show that u^λ solves the same equation for all $y \in \mathbb{R}^n$ and $s > 0$.

11. Let $(M_t)_{t \in (0,T)}$ evolve by mean curvature. For fixed $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and $\lambda > 0$ define

$$M_s^\lambda = \frac{1}{\lambda} \left(M_{\lambda^2 s + t_0} - x_0 \right).$$

Show that for each $\lambda > 0$ this gives a solution of mean curvature flow. On which time interval is this solution defined? *This exercise is related to the second exercise and also the previous one.*

12. For a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ and smooth function f defined in an open neighbourhood of M prove the identity

$$\Delta_M f = \Delta_{\mathbb{R}^{n+1}} f - H Df \cdot \nu - D^2 f(\nu, \nu).$$

Use exercise 6.

13. Let $(M_t)_{t \in (0,T)}$ evolve by mean curvature. Suppose that each M_t is of the form $M_t = \{x \in \mathbb{R}^{n+1}, w(x,t) = 0\}$ with $Dw(x,t) \neq 0$ for all $x \in M_t$ for some smooth function $w_t = w(\cdot, t) : \mathbb{R}^{n+1} \to \mathbb{R}$ and let $\Omega_t = \{x \in \mathbb{R}^{n+1}, w(x,t) > 0\}$ be a domain enclosed by M_t that is $M_t = \partial \Omega_t$.

(a) Derive a PDE for w. Use exercise 7.

(b) Suppose that in the situation above $H_{M_t} > 0$ for all $t \in (0,T)$ where we consider the mean curvature with respect to the normal pointing out of Ω_t. Show that then $\Omega_{t_2} \subset \Omega_{t_1}$ for any $0 < t_1 < t_2 < T$.

14. Prove the product and chain rule for the heat operator as stated in Lemma 2.4 of [E].

15. Let $\eta \in C_0^2(U)$ for an open set $U \subset \mathbb{R}^{n+1}$. Suppose also that $\eta \geq 0$. Prove that then

$$\frac{|D\eta(x)|^2}{\eta(x)} \leq 2 \max_U |D^2 \eta|$$

holds for all $x \in U$ with $\eta(x) > 0$.

Hint: Use that the left hand side equals

$$|D\sqrt{\eta(x)}|^2$$

for all $x \in U$ with $\eta(x) > 0$. You may want to study maximum points of the function

$$|D\sqrt{\eta + \epsilon^2}|^2$$

for arbitrary $\epsilon > 0$ and then let $\epsilon \searrow 0$. As an alternative you could use Taylor's formula with remainder in an appropriate fashion.

16. Show that the function defined by $\phi(x,t) = t((\rho^2 - |x|^2)_+)^p$ is a localization function (as in Definition 3.15 of [E]) for $p > 2$ and $t \in (0, t_0)$.

17. Show that any localization function ϕ satisfies the estimate

$$\left|\left(\frac{d}{dt} - \Delta_{M_t}\right)\phi\right| \leq c(n, c_\phi)$$

if (M_t) is a smooth solution of mean curvature flow.

18. Let M be a compact Riemannian manifold. Let $f : M \times (0,T) \to \mathbb{R}$ be smooth, continuous up to $t = 0$ and satisfy

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \geq (\leq)\, bf$$

where $b : M \times [0,T) \to \mathbb{R}$ is bounded. Show that if $f \geq (\leq)\, 0$ at time 0 then $f(t) \geq (\leq)\, 0$ for all $t \in (0,T)$.
Hint: It suffices to treat the case with '$\geq$'. The other case follows by replacing f with $-f$. Consider first the function $h = f + \epsilon(t + \delta)$ for $\epsilon > 0$ and $\delta > 0$. Show that h cannot reach zero on a time interval depending on δ and $\sup_M |b|$. Then let ϵ tend to zero to conclude that $f \geq 0$ on this interval. Now argue that the conclusion for f holds on all of $(0,T)$.

19. Let M be a compact Riemannian manifold. Let $f : M \times (0,T) \to \mathbb{R}$ be smooth, continuous up to $t = 0$ and satisfy

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \geq (\leq)\, bf$$

where $b : M \times [0,T) \to \mathbb{R}$ is bounded. Suppose also (as in exercise 18) that $f \geq (\leq)\, 0$ at time 0. Assume in addition that $b \geq 0$ on $M \times [0,T)$. Show that then

$$\min_M f(t) (\max_M f(t)) \geq (\leq) \min_M f(0) (\max_M f(0))$$

for all $t \in (0,T)$.
Hint: First use exercise 18 and then follow the proof of the weak maximum principle in [E], Appendix B, which corresponds to the statement of exercise 18 for the case $b = 0$.

20. Let f be as in exercise 18 in terms of the differentiabilty conditions but instead satisfy the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \leq -\delta f^p + K$$

for some $\delta > 0$, $K < \infty$ and $p > 0$. Assume also that $f \geq 0$. Prove that

(a) $$\max_M f(t) \le C(\delta, K, p, \max_M f(0))$$

and if $p > 1$

(b) $$\max_M f(t) \le C(\delta, K, p)\left(1 + t^{-\frac{1}{p-1}}\right)$$

hold where in the second estimate the constant is independent of $\max_M f(0)$.

Hint: To show (a) consider the first time the maximum of f reaches a maximal value $c > \max_M f(0)$ and then prove that c can be bounded in terms of δ, K and p. To derive (b) consider $tf(t)$.

21. Let f be as in exercise 18 in terms of the differentiabilty conditions but instead satisfy the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \ge \delta f^p$$

where $\delta > 0$ and $p > 1$, and where we assume $\min_M f(0) > 0$. Show that

$$\min_M f(t) \ge \frac{1}{(\delta(p-1))^{\frac{1}{p-1}}} \frac{1}{(t_0 - t)^{\frac{1}{p-1}}}$$

for $t \in (0, t_0)$ where

$$t_0 = \frac{1}{\delta(p-1)\min_M f(0)^{p-1}}.$$

In particular

$$\min_M f(t) \to \infty$$

for $t \nearrow t_0$.

Hint: Apply exercise 18 to $f - h$ where h solves the ODE initial value problem $h'(t) = \delta\, h(t)^p$ with $h(0) = \min_M f(0)$.

22. Show that the above statements still hold if Δ_M is replaced by a differential operator L on M which in local coordinates takes the form

$$(Lf)(p,t) = \sum_{i,j=1}^n a^{ij}(p,t)\partial_i\partial_j f(p,t) + \sum_{i=1}^n b^i(p,t)\partial_i f(p,t)$$

for bounded functions a^{ij}, b^i on $M \times (0,T)$, $1 \le i,j \le n$ where at every point $p \in M$ and for each $t \in (0,T)$ the matrix $(a_{ij}(p,t))$ is positive definite. A special case of such operators are the Laplacians on $(M, g(t))$ where $(g(t))$ is any family of evolving metrics on M. Examples are the metrics on M pulled back via the immersion maps for the hypersurfaces solving mean curvature flow (MCF) or metrics evolving by the Ricci flow. So we may in particular apply the above statements to Δ_M replaced by Δ_{M_t} for a solution (M_t) of mean curvature flow.

23. Use the evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta_{M_t}\right) H = |A|^2 H$$

as well as the appropriate preceding exercises to prove that if for a solution of mean curvature flow consisting of compact hypersurfaces we have $H > 0$ on M_0 then there is a $0 < t_0 < \infty$ such that

$$\min_{M_t} H^2 \geq \frac{n}{2} \frac{1}{t_0 - t}$$

for all $t \in (0, t_0)$.

24. Use this and the identity

$$\frac{d}{dt} \mathcal{H}^n(M_t) = -\int_{M_t} H^2$$

to prove that under the above assumptions on our solution we have

$$\mathcal{H}^n(M_t) \leq \mathcal{H}^n(M_0) \left(\frac{t_0 - t}{t_0} \right)^{\frac{n}{2}}.$$

25. Revisit problem 3 (c) using the information obtained in problems 13, 23 and 24.

26. For a smooth function f on a Riemannian manifold M prove the *Bochner formula*

$$\Delta_M |\nabla^M f|^2 = 2\, |\mathrm{Hess}_M f|^2 + 2\, \langle \nabla^M f, \nabla^M (\Delta_M f) \rangle + 2\, \mathrm{Ric}^M (\nabla^M f, \nabla^M f).$$

Here the *Ricci tensor* of M is given by $\mathrm{Ric}^M = (R^M_{ij})$ where $R^M_{ij} = R^M_{ikjk}$ and the sign convention for the Riemann tensor R^M_{ijkl} in our case is stated in Appendix A in [E].

27. For a smooth solution f of the heat equation on a compact Riemannian manifold M with $\mathrm{Ric}_M \geq 0$ which is continuous up to time 0 show that there exists a constant C depending only on $\max_M f(0)^2$ such that

$$\max_M |\nabla^M f(t)|^2 \leq \frac{C}{t}.$$

Hint: Apply the weak maximum principle in combination with the Bochner formula to $h = t|\nabla^M f|^2 + \lambda f^2$ for a suitable $\lambda > 0$.

28. For a smooth solution f of the heat equation which is continuous up to time 0 on a compact Riemannian manifold M (this time no condition on the Ricci curvature of M is required) prove

$$\frac{d}{dt} \bar{f} = 0$$

where

$$\bar{f}(t) \equiv= \frac{1}{\mathrm{vol}(M)} \int_M f(t).$$

Show furthermore that there exists a constant $\epsilon > 0$ such that

$$\frac{d}{dt} \int_M (f - \bar{f})^2 \leq -\epsilon \int_M (f - \bar{f})^2$$

and therefore $f(t) \to \bar{f}(0)$ in $L^2(M)$ exponentially in t.

Hint: Use the Poincaré inequality, namely the existence of a constant $\delta = \delta(M) > 0$ such that for all $u \in C^1(M)$

$$\delta \int_M (u - \bar{u})^2 \leq \int_M |\nabla^M u|^2.$$

29. Let $M \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and let $x_0 \in M$. Suppose that there are constants $c_0 > 0$ and $\rho > 0$ such that

$$\sup_{M \cap B_\rho(x_0)} |A|^2 \leq \frac{c_0}{\rho^2}.$$

Show that there is a constant $\theta \in (0,1)$ depending only on c_0 (and possibly on n) such that $M \cap B_{\theta\rho}(x_0)$ can be written as graph of a smooth function over a suitable neighbourhood of x_0 inside $T_{x_0}M$.

30. Let $(M_t)_{t \in (t_0 - \rho^2, t_0)}$ be a smooth solution of mean curvature flow inside $B_\rho(x_0)$ (that is all M_t are smooth hypersurfaces without boundary inside this ball) and suppose that x_0 is reached by this solution at time t_0. Assume that for all integers $m \geq 0$ there are constants c_m such that

$$\sup_{t \in (t_0 - \rho^2, t_0)} \sup_{M_t \cap B_\rho(x_0)} |\nabla^m A|^2 \leq \frac{c_m}{\rho^{2(m+1)}}.$$

Show that then there is constant $\epsilon > 0$ and a smooth hypersurface M_{t_0} inside $B_\epsilon(x_0)$ such that $M_t \to M_{t_0}$ for $t \nearrow t_0$ smoothly in compact subsets of $B_\epsilon(x_0)$.

31. Derive the evolution equation for $|A|^2$ stated in Appendix B, (B.9) of [E].

32. Derive a geometrically natural expression for

$$\frac{\partial}{\partial t} \Gamma_{ij}^k$$

for a solution of mean curvature flow that is one which depends on geometric quantities and their covariant derivatives for the evolving hypersurfaces. You may want to use normal coordinates at the point and time you are calculating. Show first that the time derivative of the Christoffelsymbols is a tensor in contrast to the Christoffelsymbols themselves. Use the fact that the difference of two connections is tensorial in all its arguments.

33. Calculate the heat operator on mean curvature flow of $\nabla_k^M A_{ij}$. Express this in terms of A_{ij} and its covariant derivatives. You will need your solution of exercise 22 for this. You may also try to derive the evolution equation for the higher order covariant derivatives of A_{ij} to get some practice.

34. For the rescaled solution (M_s^λ) in exercise 8 show that the tensor of m^{th} order covariant derivatives $\nabla^m A_\lambda$ of the second fundamental form relates to the corresponding tensor $\nabla^m A$ for the unscaled solution (M_t) as follows:

$$|\nabla^m A_\lambda(y)|^2 = \lambda^{2(m+1)} |\nabla^m A(x)|^2$$

for any natural number $m \geq 0$. Here $x \in M_t$ and $x = \lambda y + x_0$ and $t = \lambda^2 s + t_0$ that is $y \in M_s^\lambda$.

35. Extend the proof of the smoothness estimates of Proposition 3.22 of [E] to derivatives of order $m \geq 2$. Use induction on m.

36. For any smooth tensorfield $T = (T_{ij})$ prove Kato's inequality

$$|\nabla |T||^2 \leq |\nabla T|^2$$

at the points where T does not vanish (so that no differentiablity issues arise). Is symmetry of T needed? Extend the statement to tensorfields of arbitrary order and prove it.

Hint: Use the Cauchy-Schwarz inequality.

37. (a) Let M be a C^1 - hypersurface in $\mathbb{R}^{n+1}$ and $x_0 \in M$. For $\lambda > 0$ define

$$M^\lambda = \frac{1}{\lambda}(M - x_0).$$

Show that $M^\lambda \to T_{x_0}M$, the tangent space of M at x_0, as $\lambda \searrow 0$ smoothly on compact subsets of $\mathbb{R}^{n+1}$. Conclude that

$$\Theta(M, x_0) \equiv \lim_{\rho \searrow 0} \frac{\mathcal{H}^n(M \cap B_\rho(x_0))}{\omega_n \rho^n} = 1.$$

The quantity $\Theta(M, x_0)$ is called the n - *dimensional density* of M at x_0.

(b) For a family of hypersurfaces $(M_t)_{t<t_0}$ (not necessarily a solution of mean curvature flow) which is smooth near (x_0, t_0) for $x_0 \in M_{t_0}$ consider the rescaled family (M_s^λ) as in exercise 8. Prove that for every $s < 0$

$$\lim_{\lambda \searrow 0} M_s^\lambda = T_{x_0} M_{t_0},$$

the tangent space of M_{t_0} at x_0.

Hint: Write the original family locally as graph of a function near (x_0, t_0) and use the rescaling expression for the graph function in exercise 7.

(c) In the case that $\mathcal{M} = (M_t)_{t<t_0}$ evolves by mean curvature conclude from (b) that the *Gaussian density* of $\mathcal{M}$ satisfies $\Theta(\mathcal{M}, x_0, t_0) = 1$.

(d) Suppose that $(M_t)_{t<t_0}$ is smooth near (x_0, t_0) and that $f \geq 0$ satisfies

$$\left(\frac{d}{dt} - \Delta\right) f \leq 0$$

and is continuous at (x_0, t_0). Show that then

$$f(x_0, t_0) \leq \lim_{t \nearrow t_0} \int_{M_t} f \Phi_{(x_0, t_0)}.$$

38. Let $M_t = \partial B^{n+1-k}_{r(t)} \times \mathbb{R}^k$ for $r(t) = \sqrt{-2(n-k)t}$. This defines a solution $\mathcal{M}$ of mean curvature flow given by generalized cylinders which homothetically shrink to $0 \in \mathbb{R}^{n+1}$ at time 0. Show that $\Theta(\mathcal{M}, 0, 0) = c(n, k) > 1$.

Hint: The constant should be

$$c(n,k) = \left(\frac{n-k}{2\pi e}\right)^{\frac{n-k}{2}} \mathcal{H}^{n-k}(S^{n-k}).$$

If the calculations get too involved verify the statement at least for $n = 2$.

39. (a) Let $(g_{ij}(t))$ be a time-dependent family of symmetric $n \times n$ - matrices. Prove

$$\frac{d}{dt}\sqrt{\det(g_{ij}(t))} = \frac{1}{2}\sqrt{\det(g_{ij}(t))}g^{ij}\frac{d}{dt}g_{ij}.$$

(b) Let (M_t) evolve by $\frac{\partial x}{\partial t} = X$. Show that then the induced metric $(g_{ij}(t))$ evolves by

$$\frac{\partial}{\partial t}\sqrt{\det(g_{ij}(t))} = \operatorname{div}_{M_t} X \sqrt{\det(g_{ij}(t))}.$$

40. Show that $\Phi_{(x_0,t_0)}$ defined by

$$\Phi_{(x_0,t_0)}(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$$

for $x \in \mathbb{R}^{n+1}$ and $t < t_0$ satisfies the identity

$$\frac{\partial \Phi_{(x_0,t_0)}}{\partial t} + \operatorname{div}_{M_t} D\Phi_{(x_0,t_0)} + \frac{|\nabla^\perp \Phi_{(x_0,t_0)}|^2}{\Phi_{(x_0,t_0)}} = 0.$$

REFERENCES

[E] K. Ecker *Regularity Theory for Mean Curvature Flow*, Birkhäuser 2004