Fachbereich Mathematik und Informatik
Freie Universität Berlin

Einführung in die Differentialgeometrie WS 2018/2019

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Sheet 4

Please use the formula from lectures

$$H(x) = -\text{div}\, N(x) = \text{div}\left(\frac{\nabla f}{|\nabla f|}\right)(x)$$

for the mean curvature of a (level) n - surface $S = f^{-1}(c)$ with smooth $f : \Omega \subset \mathbf{R}^{n+1} \to \mathbf{R}$. We chose $N = -\frac{\nabla f}{|\nabla f|}$ as our unit normal field at the point $x \in S$. Here the divergence of a vector field $X : \Omega \subset \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is defined by

$$\text{div} X = \sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_k}.$$

Problem 1. Show that the mean curvature of the catenoid $S = \{x \in \mathbf{R}^3, \cosh x_3 = \sqrt{x_1^2 + x_2^2}\}$ vanishes everywhere.

3 points

Problem 2. Let $S = g^{-1}(0)$ be the n - surface of revolution of problem 2, sheet 1. Consider in particular the special case $f(t, r) = u(t) - r$ in problem 3, sheet 1.

(a) Show: The mean curvature of S w.r.t. $N = -\frac{\nabla g}{|\nabla g|}$ at $x \in S$ is given by the formula

$$H(x) = \frac{u''(x_{n+1})}{(1 + u'(x_{n+1})^2)^{3/2}} - \frac{n-1}{r\sqrt{1 + u'(x_{n+1})^2}}$$

where $r = \sqrt{x_1^2 + \ldots x_n^2} = u(x_{n+1})$.

(b) Use this formula to check your result of problem 1.

3 points

Problem 3. Let $S = \{x \in \mathbf{R}^{n+1}, f(x) = c\}$ be an n - (level) surface. For $\lambda > 0$, define the function $f_\lambda : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f_\lambda(y) = f(\frac{y}{\lambda})$. Let $S_\lambda = \{y \in \mathbf{R}^{n+1}, f_\lambda(y) = c\}$.
Show:

(a) $S_\lambda = \lambda S \equiv \{\lambda x, x \in S\}$.

(b) S_λ is a (level) n - surface.

(c) The mean curvature H_λ of S_λ satisfies

$$H_\lambda(\lambda x) = \frac{1}{\lambda} H(x)$$

for all $x \in S$ where $H(x)$ is the mean curvature of S at x.

4 points

To be returned on Tuesday 27. November 2018 at 10 a.m. (sharp) to Klaus Ecker's tutorial box