Fachbereich Mathematik und Informatik
Freie Universität Berlin

Einführung in die Differentialgeometrie WS 2018/2019

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Sheet 5

Problem 1. Let $F(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$ for $a > b > 0$ and $0 < \theta < 2\pi, 0 < \phi < 2\pi$ be a parametrization of the torus T in $\mathbf{R}^3$.

(a) Compute $Area\,(T)$.

(b) Compute $\int_T K\, dA$ where K is the Gauß curvature of T .

2 points

Problem 2. Let $A = (a_{ij})$ be a symmetric, invertible $n \times n$ - matrix and let $A^{-1} = (a^{ij})$ denote its inverse.

(a) Show: $\frac{\partial}{\partial a_{ij}} \det A = \det A \cdot a^{ij}$ for all $1 \le i, j \le n$.

(b) Consider now a family of $n \times n$ - matrices $(A(t)) = (a_{ij}(t))$, $t \in I$ for an open interval $I \subset \mathbf{R}$. We furthermore assume that $a_{ij} : I \to \mathbf{R}$ is differentiable for all $1 \le i, j \le n$. Show: $\frac{d}{dt} \det A = \det A \sum_{i,j=1}^{n} a^{ij} \frac{d}{dt} a_{ij}$ in I.

2 points

Problem 3. Let $S = F(U)$ be a smooth parametrized n - surface in $\mathbf{R}^{n+1}$. Let $\eta : U \to \mathbf{R}$ be a fixed but arbitrary smooth function with compact support, that is the closure of the set $\{u \in U,\, \eta(u) \neq 0\}$ is a compact subset of the open set U. (This is a stronger than assuming that η vanishes on the boundary of U! After all U could be unbounded. If $U$is bounded such functions vanish in a whole neighbourhood of the boundary of U.) Let N be the standard unit normal field for S. For $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$ define $F_t(u) = F(u, t) = F(u) + t\,\eta(u)\,N(u)$. Let $g_{ij}(u, t) = \frac{\partial F_t(u)}{\partial u_i} \cdot \frac{\partial F_t(u)}{\partial u_j}$.

(a) Show: For fixed η as above there is an $\epsilon > 0$ such that the sets $S_t = F_t(U)$, $t \in (-\epsilon, \epsilon)$ are smooth parametrized n - surfaces.

(b) Show: $\frac{d}{dt}_{|t=0} \sqrt{\det(g_{ij}(u,t))} = -\sqrt{\det(g_{ij}(u))}\,\eta(u)\,H(u)$ for all $u \in U$. H is the mean curvature function of S. (Hint: Use problem 2 (b).)

(c) Let S be a minimal surface that is for each fixed η as above and associated $\epsilon > 0$ (as in part (a)) we have $Area\,(S) \le Area\,(S_t)$ for all $t \in (-\epsilon, \epsilon)$. Conclude from this that $\int_S H\,\eta\, dA = 0$ for all smooth $\eta : U \to \mathbf{R}$ with compact support.

(d) By choosing a suitable η in (c) show that H vanishes everywhere.

6 Punkte

To be returned on Tuesday 11 December 2018 by 10 a.m. (sharp) to Klaus Ecker's tutorial box.