Fachbereich Mathematik und Informatik
Freie Universität Berlin

Einführung in die Differentialgeometrie WS 2018/2019

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Sheet 6

Problem 1. Let $S = F(U)$ and $\widetilde{S} = \widetilde{F}(U)$ for $U \subset \mathbf{R}^n$ be smooth parametrized n - surfaces in $\mathbf{R}^{n+1}$ and $\psi : S \to \widetilde{S}$ be a map with $\widetilde{F} = \psi \circ F$. Such a map ψ is called a *local isometry* between S and $\widetilde{S}$ if the metrics $(g_{ij}(u))$ und $(\tilde{g}_{ij}(u))$ associated to F and $\widetilde{F}$ are equal for every $u \in U$. If in addition ψ is also bijective (one-one and onto) we call ψ a *(global) isometry* between S and $\widetilde{S}$.

Let $C_r = \{x \in \mathbf{R}^3,\, x_1^2 + x_2^2 = r^2\}$ be the cylinder of radius $r > 0$. Show: The map $\psi : C_2 \to C_1$ defined by $\psi(2\cos\theta, 2\sin\theta, z) = (\cos 2\theta, \sin 2\theta, z)$ is a local but not a global isometry.

4 points

Problem 2. The *Christoffelsymbols* Γ^k_{ij}, $1 \le i, j, k \le n$ of a smooth parametrized n - surface $S = F(U)$ are defined by

$$\left(\frac{\partial^2 F}{\partial u_i \partial u_j}\right)^T = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial F}{\partial u_k}.$$

Here $X^T \equiv X - (X \cdot N)\, N$ is the projection of the vector field X onto the tangent spaces of S. Show:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial}{\partial u_i} g_{jl} + \frac{\partial}{\partial u_j} g_{il} - \frac{\partial}{\partial u_l} g_{ij}\right).$$

2 points

Problem 3. We write $\nabla_{\frac{\partial F}{\partial u_i}} X = (\frac{\partial X}{\partial u_i})^T$ for the covariant derivative of a vector field X in the direction of $\frac{\partial F}{\partial u_i}$. The components R_{ijkl} of the *curvature tensor* $(1 \le i, j, k, l \le n)$ of a smooth parametrized n - surface $S = F(U)$ are defined by

$$R_{ijkl} = \left(\nabla_{\frac{\partial F}{\partial u_j}} \nabla_{\frac{\partial F}{\partial u_i}} \frac{\partial F}{\partial u_k}\right) \cdot \frac{\partial F}{\partial u_l} - \left(\nabla_{\frac{\partial F}{\partial x_i}} \nabla_{\frac{\partial F}{\partial u_j}} \frac{\partial F}{\partial u_k}\right) \cdot \frac{\partial F}{\partial u_l}.$$

Derive the Gauß equations

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}.$$

Here $h_{ij} = -\frac{\partial F}{\partial u_i} \cdot \frac{\partial N}{\partial u_j} = \frac{\partial^2 F}{\partial u_i \partial u_j} \cdot N$. Show in particular: For $n = 2$ the *theorema egregium*

$$R_{1212} = K(g_{11} g_{22} - g_{12}^2)$$

holds where K ist die Gauß curvature of S.

4 points

To be returned on Tuesday 18. December 2018 at 10 a.m.(sharp) to Klaus Ecker's tutorial box