Fachbereich Mathematik und Informatik
Freie Universität Berlin

Einführung in die Differentialgeometrie WS 2018/2019

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Sheet 8

Problem 1. A differentiable manifold M is called *orientable* if it admits a differentiable atlas $\{(x_\alpha, U_\alpha)\}$ such that for every $p \in U_\alpha \cap U_\beta$ we have

$$\det\left(D(x_\alpha \circ x_\beta)(x_\beta(p))\right) > 0.$$

Such an atlas is called an *orientation* for M.

(a) Let $\{(x_1, U_1), (x_2, U_2)\}$ be a differentiable atlas for M such that $U_1 \cap U_2$ is connected in M. Show that then M is orientable.

(b) Use problem 3, sheet 7, to find an explicit orientation for S^n (Hint: On sheet 7 you had to prove that $x_S \circ x_N^{-1}$ and $x_N \circ x_S^{-1}$ are differentiable in their respective domains. Here you may need to compute their derivatives explicitely.) **6 points**

Problem 2. As defined in lectures, a *derivation* on a smooth manifold M is a map $A : C^\infty(M) \to C^\infty(M)$ satisfying (i) $A(af + bg) = aA(f) + bA(g)$ for all $a, b \in \mathbf{R}$, $f, g \in C^\infty(M)$ and (ii) $A(fg) = fA(g) + A(f)g$ for all $f, g \in C^\infty(M)$. For a smooth vector field X on a differentiable manifold M we define (as in lectures) the associated *derivation* $\bar{X}$ by $\bar{X}(f)(p) = X(p)(f)$ for all $f \in C^\infty(M)$ and all $p \in M$. Let X, Y be smooth vector fields on a differentiable manifold M and let $\bar{X}, \bar{Y}$ be their associated derivations.

(a) Show: The expression XY defined in the canonical way by composition of the associated derivations is in general not a vector field. (Hint: Show that $\bar{X}\bar{Y}$ generally does not satisfy the product rule (ii) above.)

(b) Show: The *Lie bracket* defined by $[X, Y] = XY - YX$ (again using $\bar{X}$ and $\bar{Y}$) is a vector field, that is gives rise to a derivation on M. **2 points**

Problem 3. Derive the following properties of the Lie bracket on a smooth manifold M:

(a)

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all smooth vector fields X, Y, Z. This is the so-called *Jacobi-identity.*

(b) Derive a formula for $[fX, gY]$ for vector fields X, Y and $f, g \in C^\infty(M)$ and use this to show that

$$[fX, Y] \neq f[X, Y]$$

in general. **2 points**

To be returned on Monday 28. January 2019 at 12 noon (sharp) to Klaus Ecker's tutorial box