**Fachbereich Mathematik und Informatik**
**Freie Universität Berlin**

**Einführung in die Differentialgeometrie WS 2018/2019**

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**Sheet 9**

**Problem 1.** Let $M$ be an n- dimensional differentiable manifold with a Riemannian metric $g$. A map $\phi : M \to M$ is called an *isometry* if it is a diffeomorphism and

$$g(\phi(p))(d\phi(p)(v_p), d\phi(p)(w_p)) = g(p)(v_p, w_p)$$

holds for all $v_p, w_p \in T_pM$.

Let $\mathbf{H}^n$ be *hyperbolic space* defined as the manifold $U = \{p \in \mathbf{R}^n, p_n > 0\}$ with metric given by $g = (g_{ij})$ with $g_{ij}(p) = \frac{1}{(p_n)^2}\delta_{ij}$, $1 \leq i, j \leq n$. Show that the following maps $\phi : \mathbf{H}^n \to \mathbf{H}^n$ are isometries.

(a) $\phi(p) = (\psi(p_1, \ldots, p_{n-1}), p_n)$, $p \in U$, for an arbitrary isometry $\psi$ of $\mathbf{R}^{n-1}$.

(b) $\phi(p) = \lambda p$, $p \in U$, $\lambda > 0$ fixed.

(c) $\phi(p) = (p_1, \ldots, -p_i, \ldots, p_n)$ für $1 \leq i \leq n-1$

(d) $\phi(p) = \frac{p}{|p|^2}$, $|p| = \sqrt{\sum_{i=1}^n (p_i)^2}$.

Which of these maps are isometries of $\mathbf{R}^n$ with the standard metric and which are not. Justify your answers.

**6 points**

**Problem 2.** Let $D$ denote the standard connection on $\mathbf{R}^3$. Show that

$$\nabla_X Y := D_X Y + \frac{1}{2} X \times Y$$

defines an affine connection on $\mathbf{R}^3$. Here $\times$ denotes the usual cross-product of vectors in $\mathbf{R}^3$.

**1 point**

**Problem 3.** Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. Define

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}\big(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle\big)$$

for smooth vector fields $X, Y, Z$ auf $M$. Show that $\nabla$ defines a Riemannian connection i.e. $\nabla$ is an affine connection with the additional properties $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ and $[X, Y] = \nabla_X Y - \nabla_Y X$ for smooth vector fields $X, Y, Z$.

**3 points**

To be returned on Monday 4. February 2019 at 12 noon (sharp) to Klaus Ecker's tutorial box