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Einführung in die Differentialgeometrie WiSe 2018/19

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Sheet 10

Problem 1. We define the component functions of the curvature tensor as in lectures by

$$R_{ijkl} = \langle R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i})\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\rangle$$

where $1 \le i,j,k,l \le n$. For a vector field Z on M set

$$\nabla_i \nabla_j Z_k = \langle \nabla^2_{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}} Z, \frac{\partial}{\partial x^k}\rangle$$

where $\nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$ is the Hessian operator. Show that

$$\nabla_i \nabla_j Z_k - \nabla_j \nabla_i Z_k = \sum_{l=1}^{n} R_{ijkl} Z^l$$

where Z^l is the l^{th} component of the vector field Z with respect to the coordinate basis vector fields given by the $\frac{\partial}{\partial x^i}$ for $1 \le i \le n$.

Problem 2. The component functions R_{ij} of the *Ricci tensor* are defined by

$$R_{ij} = \sum_{k,l} g^{kl} R_{ikjl}.$$

Show that $R_{ij} = R_{ji}$ for all i and j.

Problem 3. Let $R = \sum_{i,j} g^{ij} R_{ij}$ be the scalar curvature. We define the component functions W_{ijkl} of the *Weyl tensor* on a Riemannian manifold M (of dimension at least equal to 3) by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) + \frac{1}{(n-1)(n-2)} R(g_{ik}g_{jl} - g_{il}g_{jk}).$$

(a) Prove that W_{ijkl} has the same symmetry properties as R_{ijkl} (including the first Bianchi identity). In particular $W_{ijkl} = 0$ if at least three of the indices $1 \le i,j,k,l \le n$ are equal.
(b) Show that in addition the identity $\sum_k g^{ik} W_{ijkl} = 0$ holds for all j, l.
(c) Use (a) and (b) to show that $W_{ijkl} = 0$ if M has dimension equal to 3. (Hint: Prove that $W_{1212} = W_{1313} = W_{2323} = 0$).

This sheet **will no**t be counted towards your active participation in the tutorials. However, as the curvature tensor was covered in lectures you ought to be able to solve the above problems and take the opportunity to use this sheet for lecture revision and exam preparation. The topics on this sheet are certainly examinable.